

Math 151
Section 2.2

Derivatives of polynomials

Review of the limit definition of a derivative.

| | |
|----------------------|------------------------|
| $f(x)$ | $f'(x)$ |
| $f(x) = 4x + 5$ | $f'(x) = 4$ |
| $f(x) = x^2 - 2x$ | $f'(x) = 2x - 2$ |
| $f(x) = x^3 + 4$ | $f'(x) = 3x^2$ |
| $f(x) = 2x^4 - 4x^3$ | $f'(x) = 8x^3 - 12x^2$ |

Power Rule

If $f(x) = cx^n$, then $f'(x) = ncx^{n-1}$

Constant Rule

If $f(x) = C$ where C is a constant, then $f'(x) = 0$

Proof of the power rule

Let $f(x) = x^n$, and find $f'(x)$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n \right] - x^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}xn^{n-2}h^2 + \dots + h^n}{h} \\
 &= nx^{n-1}h + 0 + 0 + \dots + 0 \\
 &= nx^{n-1}
 \end{aligned}$$

Examples using the power rule.

Example 1

If $f(x) = 2x^3$, then find $f'(x)$ using the power rule.

$$f'(x) = 3 \cdot 2x^{3-1} = 6x^2$$

Example 2

If $f(x) = x^4 + 2x^3 + 3x$, then find $f'(x)$ using the power rule.

$$f'(x) = 4x^{4-1} + 3 \cdot 2x^{3-1} + 3 = 4x^3 + 6x^2 + 3$$

Example 3

If $f(x) = \frac{1}{x^2}$, then find $f'(x)$ using the power rule.

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$f'(x) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$$

Example 4

If $f(x) = \frac{1}{x^3} + \frac{4}{x^2}$, then find $f'(x)$ using the power rule.

$$f(x) = \frac{1}{x^3} + \frac{4}{x^2} = x^{-3} + 4x^{-2}$$

$$f'(x) = -3x^{-3-1} + (-2)4x^{-2-1} = -3x^{-4} - 8x^{-3} = -\frac{3}{x^4} - \frac{8}{x^3}$$

Example 5

If $f(x) = \sqrt{x}$, then find $f'(x)$ using the power rule.

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

Example 6

If $f(x) = 6\sqrt{x}$, then find $f'(x)$ using the power rule.

$$f(x) = 6\sqrt{x} = 6x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(6)x^{\frac{1}{2}-1} = 3x^{-\frac{1}{2}} = \frac{3}{x^{\frac{1}{2}}} = \frac{3}{\sqrt{x}}$$

Example 7

If $f(x) = 2\sqrt[3]{x}$, then find $f'(x)$ using the power rule.

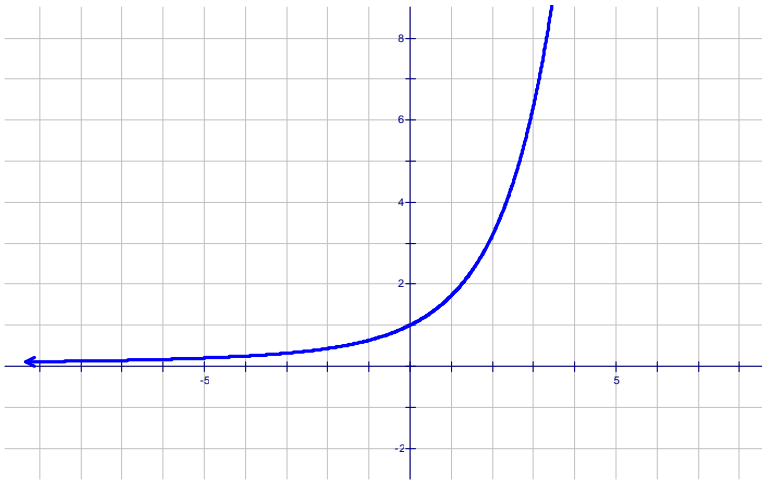
$$f(x) = 2\sqrt[3]{x} = 2x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3} \cdot 2x^{\frac{1}{3}-1} = \frac{2}{3}x^{-\frac{2}{3}} = \frac{2}{3x^{\frac{2}{3}}} = \frac{2}{3\sqrt[3]{x^2}}$$

The exponential function

The definition of the number e

e is the number such that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$



The derivative of the exponential function

$$f(x) = e^x$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h} = \lim_{h \rightarrow 0} \frac{e^x (e^h - 1)}{h} \\ &= \lim_{h \rightarrow 0} e^x \left(\frac{e^h - 1}{h} \right) = e^x \left[\lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right) \right] = e^x (1) = e^x \end{aligned}$$

Definition: The derivative of $f(x) = e^x$ is $f'(x) = e^x$

Example 8

Find the derivative of $f(x) = 6e^x$

$$f'(x) = 6e^x$$

Example 9

Find the equation of a tangent line to the function $f(x) = x^2 + x$ through the point (1,2).

Find the derivative of $f(x) = x^2 + x$

$$f'(x) = 2x + 1$$

Find the slope of the tangent line: $m = f'(1) = 2(1) + 1 = 2 + 1 = 3$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 3(x - 1)$$

$$y - 2 = 3x - 3$$

$$y - 2 + 2 = 3x - 3 + 2$$

$$y = 3x - 1$$

Example 10

Find the equation of a tangent line to the function $f(x) = x^3 + e^x$ through the point (0,1).

Find the derivative of $f(x) = x^3 + e^x$

$$f'(x) = 3x^2 + e^x$$

Find the slope of the tangent line: $m = f'(0) = 3(0)^2 + e^0 = 0 + 1 = 1$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 1(x - 0)$$

$$y - 1 = x$$

$$y - 1 + 1 = x + 1$$

$$y = x + 1$$

Example 11

The equation of motion of a particle is $s(t) = t^3 - 4t$ where s is in meters and t is seconds.

a) Find the velocity and acceleration as a function of t .

$$v(t) = s'(t) = 3t^{3-1} - 4 = 3t^2 - 4$$

$$a(t) = s''(t) = 6t$$

b) Find the velocity after 2 seconds.

$$v(t) = 3t^{3-1} - 4 = 3t^2 - 4$$

$$v(2) = 3(2)^2 - 4 = 12 - 4 = 8 \frac{m}{s}$$

c) Find the acceleration after 2 seconds.

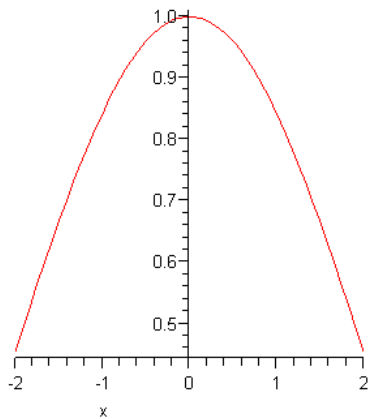
$$a(t) = s''(t) = 6t$$

$$a(2) = 6(2) = 12 \frac{m}{s^2}$$

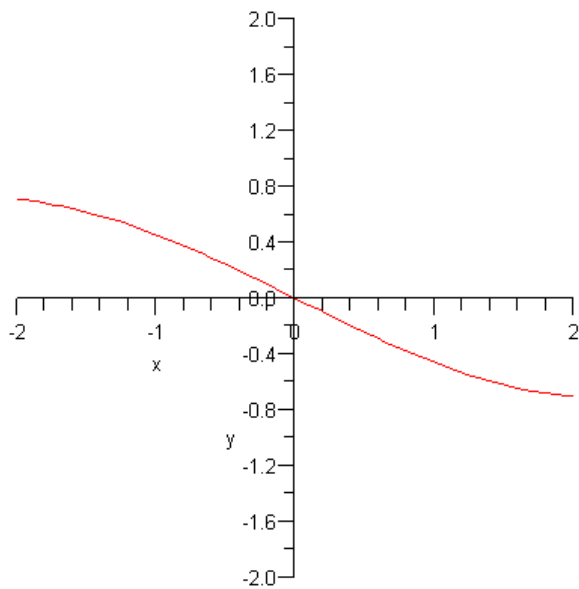
Derivatives of Trigonometric Functions

Review

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$$



$$\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0$$



The Derivative of Sine

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x)\sin(h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\ &= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\ &= \sin(x)(0) + \cos(x)(1) \\ &= \cos(x) \end{aligned}$$

Rule 1: If $f(x) = \sin(x)$, $f'(x) = \cos(x)$

The Derivative of Cosine

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} - \lim_{h \rightarrow 0} \frac{\sin(x)\sin(h)}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(x)(\cos(h) - 1)}{h} - \sin(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\&= \cos(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \sin(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\&= \cos(x)(0) - \sin(x)(1) \\&= -\sin(x)\end{aligned}$$

Rule 2: If $f(x) = \cos(x)$, $f'(x) = -\sin(x)$

The Derivative of Tangent

If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$

Example 12

Find the derivative of $y = 2x^2 + 5 \cos(x)$

$$y = 2x^2 + 5 \cos(x)$$

$$y' = \frac{d}{dx}(2x^2) + \frac{d}{dx}(5 \cos(x))$$

$$y' = 4x - 5 \sin(x)$$

Example 13

Find the derivative of $y = 2e^x + 4\sin(x)$

$$y = 2e^x + 4\sin(x)$$

Solution:

$$y' = \frac{d}{dx}(2e^x) + \frac{d}{dx}(4\sin(x))$$

$$y' = 2e^x + 4\cos(x)$$