

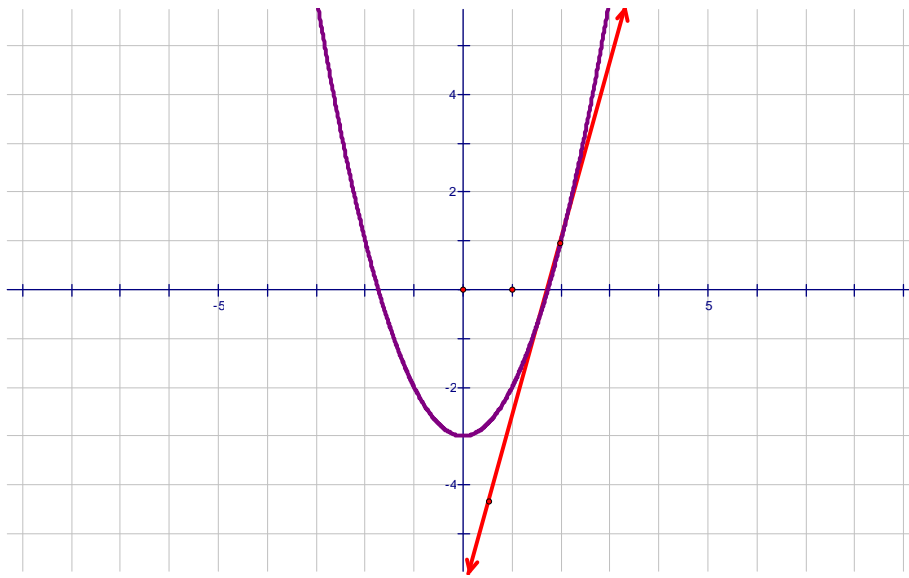
Math 151

Section 2.1

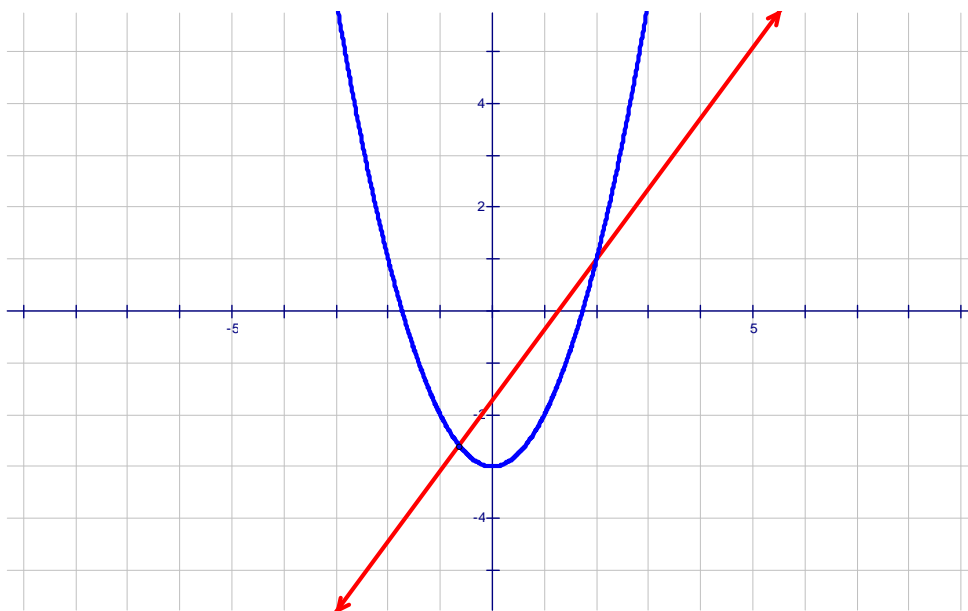
Limit Definition of a Derivative

Rate of change = slope

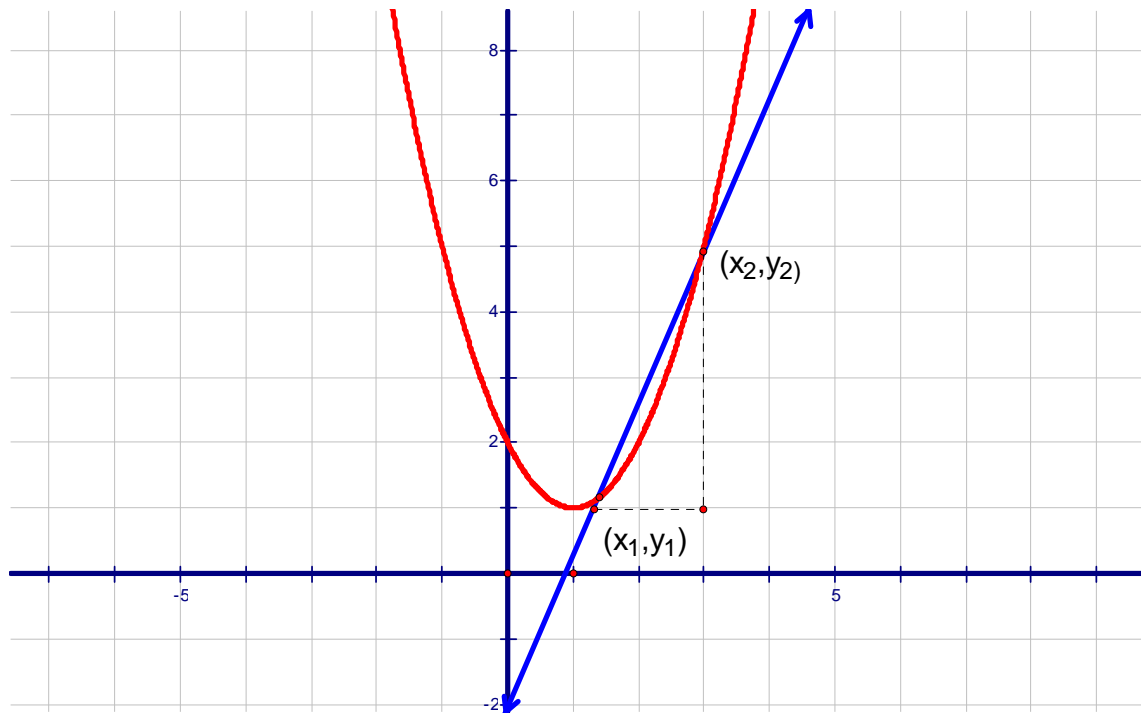
Tangent lines



Secant lines



The limit definition of a derivative



Find the slope of the tangent line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{x+h-x}$$

$$m = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Therefore, the derivative of the function is:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Example 1

Use the limit definition of a derivative to find the derivative of the function

$$f(x) = 3x + 4$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{x \rightarrow 0} \frac{3(x+h) + 4 - (3x+4)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{3x + 3h + 4 - 3x - 4}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{3h}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} 3$$

$$f'(x) = 3$$

Example 2

Given $f(x) = 5x + 3$, find $f'(x)$ using the limit definition of a derivative.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{5(x+h) + 3 - (5x+3)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{5x + 5h + 3 - 5x - 3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{5h}{h}$$

$$= \lim_{h \rightarrow 0} 5$$

$$= 5$$

Example 3

Given $f(x) = x^2 - 2$, find $f'(x)$ using the limit definition of a derivative.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 2 - (x^2 - 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)(x+h) - 2 - x^2 + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + xh + xh + h^2 - 2 - x^2 + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 2 - x^2 + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} = \lim_{h \rightarrow 0} 2x + h = 2x + 0 = 2x \end{aligned}$$

Example 4

Given $f(x) = x^2 + 5$, find $f'(x)$ using the limit definition of a derivative.

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 5 - (x^2 + 5)}{h} \\&= \lim_{h \rightarrow 0} \frac{(x+h)(x+h) + 5 - x^2 - 5}{h} \\&= \lim_{h \rightarrow 0} \frac{x^2 + xh + xh + h^2 + 5 - x^2 - 5}{h} \\&= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 + 5 - x^2 - 5}{h} \\&= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\&= \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} \\&= \lim_{h \rightarrow 0} 2x + h \\&= 2x + 0 \\&= 2x\end{aligned}$$

Example 5

Find the derivative using the limit definition of a derivative.

$$f(x) = x^2 + 3x$$

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)(x+h) + 3x + 3h - x^2 - 3x}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{x^2 + xh + xh + h^2 + 3x + 3h - x^2 - 3x}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 3h}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h + 3)}{h} = \lim_{h \rightarrow 0} 2x + h + 3 = 2x + 0 + 3 = 2x + 3 \end{aligned}$$

Example 6

Find the derivative of $f(x) = x^3 - 4$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - 4 - (x^3 - 4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)(x+h)(x+h) - 4 - x^3 + 4}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2)(x+h) - 4 - x^3 + 4}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 + 3xh^2 + 3x^2h + h^3 - 4 - x^3 + 4}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\ &= \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 = 3x^2 + 3x(0) + 0^2 = 3x^2 \end{aligned}$$

Example 7

Find the derivative of $f(x) = \frac{1}{x-4}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h-4} - \frac{1}{x-4}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h-4)(x-4)} - \frac{1}{(x-4)(x+h-4)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x-4}{(x+h-4)(x-4)} - \frac{x+h-4}{(x-4)(x+h-4)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x-4 - (x+h-4)}{(x+h-4)(x-4)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x-4-x-h+4}{(x+h-4)(x-4)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{h}{(x+h-4)(x-4)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{(x+h-4)(x-4)} \\ &= \frac{1}{(x+0-4)(x-4)} \\ &= \frac{1}{(x-4)^2} \end{aligned}$$

Example 9

Find the equation of tangent line to the curve $f(x) = x^2 - 3x$ at the point $(1, -2)$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$$

$$f'(x) = \lim_{x \rightarrow 0} \frac{(x+h)(x+h) - 3x - 3h - x^2 + 3x}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^2 + xh + xh + h^2 - 3x - 3h - x^2 + 3x}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h - 3x - 3h - x^2 + 3x}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h - 3)}{h} = \lim_{h \rightarrow 0} 2x + h - 3 = 2x + 0 - 3 = 2x - 3$$

Now find the slope

$$f'(1) = 2(1) - 3 = -1 \Rightarrow m = -1$$

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = -1(x - 1)$$

$$y + 2 = -x + 1$$

$$y = -x - 1$$

Example 10

Find the derivative of

$$f(x) = \frac{1}{x+2}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)+2} - \frac{1}{x+2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x+2}{(x+h+2)(x+2)} - \frac{x+h+2}{(x+h+2)(x+2)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{x+2 - (x+h+2)}{(x+h+2)(x+2)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{x+2 - x - h - 2}{(x+h+2)(x+2)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h}{(x+h+2)(x+2)}}{h} = \lim_{h \rightarrow 0} \frac{-h}{(x+h+2)(x+2)} \cdot \frac{1}{h} = \lim_{h \rightarrow 0} \frac{-1}{(x+h+2)(x+2)} = \frac{-1}{(x+0+2)(x+2)} \\ &= \frac{-1}{(x+2)^2} \text{ or } \frac{1}{x^2 + 4x + 4} \end{aligned}$$

Example 11Find the derivative of the function $f(x) = \sqrt{x-3}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h-3} - \sqrt{x-3}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{x+h-3} - \sqrt{x-3})(\sqrt{x+h-3} + \sqrt{x-3})}{h(\sqrt{x+h-3} + \sqrt{x-3})} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h-3})^2 - \sqrt{x+h-3}\sqrt{x-3} + \sqrt{x+h-3}\sqrt{x-3} - (\sqrt{x-3})^2}{h(\sqrt{x+h-3} + \sqrt{x-3})} = \lim_{h \rightarrow 0} \frac{x+h-3 - (x-3)}{h(\sqrt{x+h-3} + \sqrt{x-3})} \\ &= \lim_{h \rightarrow 0} \frac{x+h-3-x+3}{h(\sqrt{x+h-3} + \sqrt{x-3})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h-3} + \sqrt{x-3})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h-3} + \sqrt{x-3}} = \frac{1}{\sqrt{x+0-3} + \sqrt{x-3}} \\ &= \frac{1}{\sqrt{x-3} + \sqrt{x-3}} = \frac{1}{2\sqrt{x-3}} \end{aligned}$$
