Math 151
Extrema of a Function

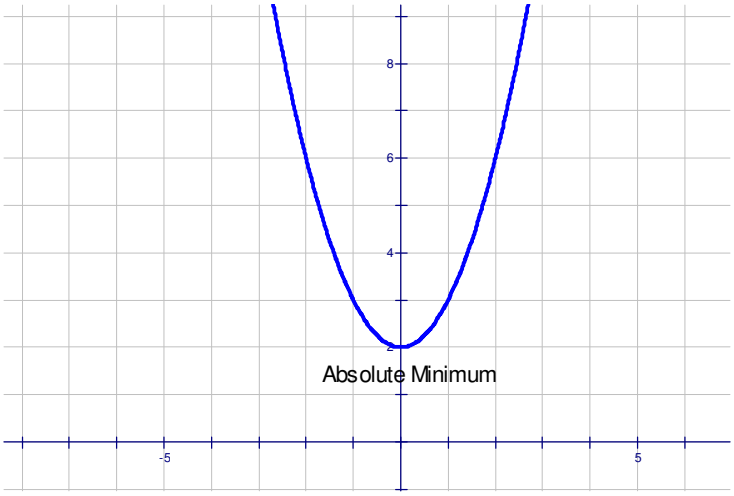
Definition of Extrema

Let f be defined on an interval I containing c.

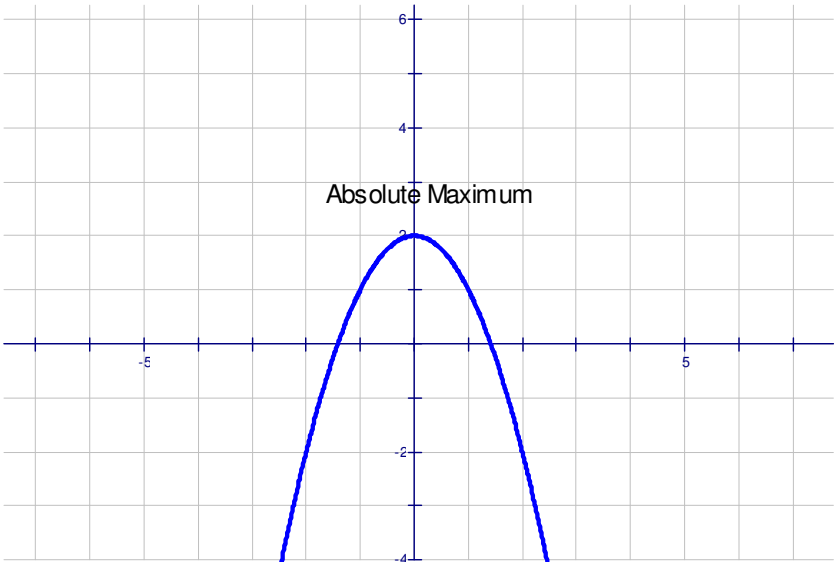
1) $f(c)$ is the minimum of f if $f(c) \leq f(x)$ for all x on I.
2) $f(c)$ is the maximum of f if $f(c) \geq f(x)$ for all x on I.

Types of extrema points

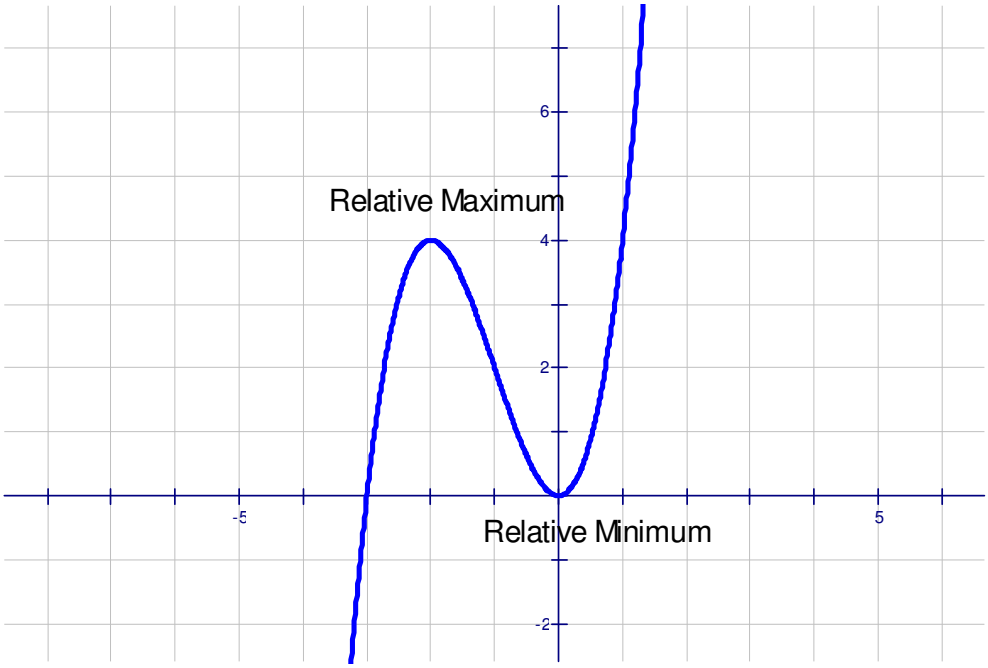
Absolute Minimum



Absolute Maximum



Relative Maximums and Relative Minimums



First Derivative Test

Let c be a critical number of the function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows:

1) If $f'(x)$ changes from negative to positive at c, then f has a relative maximum or absolute maximum at $(c, f(c))$

2) If $f'(x)$ changes from positive to negative at c, then f has a relative minimum or absolute minimum at $(c, f(c))$

3) If $f'(x)$ changes from negative to negative or positive to positive at c, then f has neither relative maximum nor relative minimum.

Example 1

Find any extrema points of the function. $f(x) = x^2 + 2x$

$f(x) = x^2 + 2x$

$f'(x) = 2x + 2$

$2x + 2 = 0$

$2x + 2 - 2 = 0 - 2$

$2x = -2$

$\frac{2x}{2} = \frac{-2}{2}$

$x = -1$

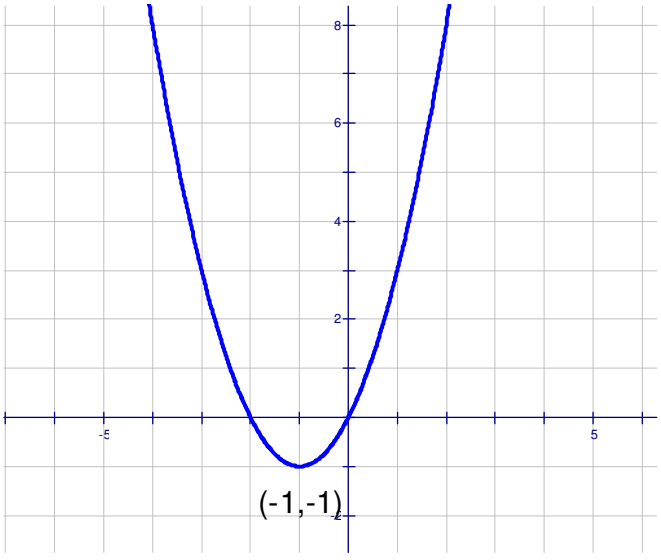
$f(-1) = (-1)^2 + 2(-1) = 1 - 2 = -1$

Critical pt. $(-1,-1)$

$f'(-2) = 2(-2) + 2 = -2$

$f'(1) = 2(1) + 2 = 4$

Interval	$(-\infty,-1)$	$(-1,\infty)$
Test Value	$x = -2$	$x = 1$
Sign of $f'(x)$	Negative	Positive
Conclusion	Decreasing	Increasing



The function is increasing when x is less than -1 and increasing when x is greater than -1.

$\Rightarrow$ *At* $(-1,-1)$, f *has an absolute* min.

Example 2

Find all extrema points of the function. $f(x) = x^3 - 1$

$f(x) = x^3 - 1$

$f'(x) = 3x^2$

$3x^2 = 0$

$\frac{3x^2}{3} = \frac{0}{3}$

$x^2 = 0$

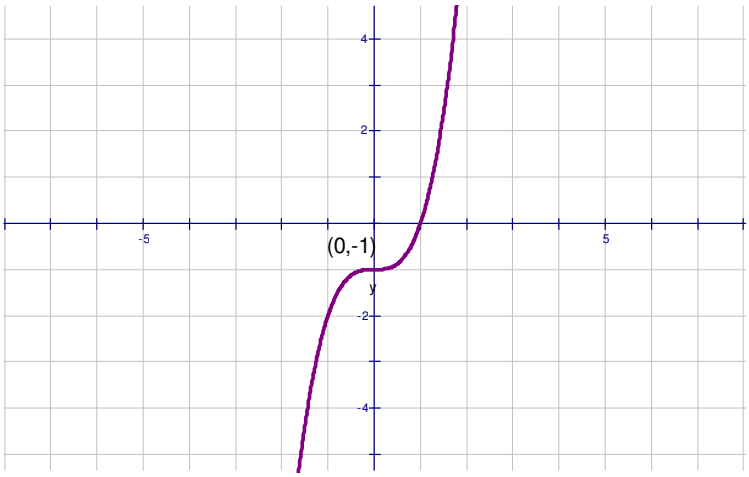
$x = 0$

$\Rightarrow f(0) = 0^3 - 1 = -1 \Rightarrow critical\ \ pt.\ is\ (0,-1)$

$f'(-1) = 3(-1)^2 = 3$

$f'(1) = 3(1)^2 = 3$

Interval	$(-\infty,0)$	$(0,\infty)$
Test Value	$x = -1$	$x = 1$
Sign of $f'(x)$	Positive	Positive
Conclusion	Increasing	Increasing



The function is increasing when x is less than zero and greater than zero, so the function has an inflection point at $(0,-1)$. (See graph above)

Example 3

Find all extrema points of the function. $f(x) = 2x^3 - 3x^2$

$f(x) = 2x^3 - 3x^2$

$f(x) = 6x^2 - 6x$

$6x^2 - 6x = 0$

$6x(x-1) = 0$

$6x = 0 \text{ or } x - 1 = 0$

$\frac{6x}{6} = \frac{0}{6} \qquad x = 1$

$x = 0$

$f(0) = 2(0)^3 - 3(0)^2 = 0 - 0 = 0$

$f(1) = 2(1)^3 - 3(1)^2 = 2 - 3 = -1$

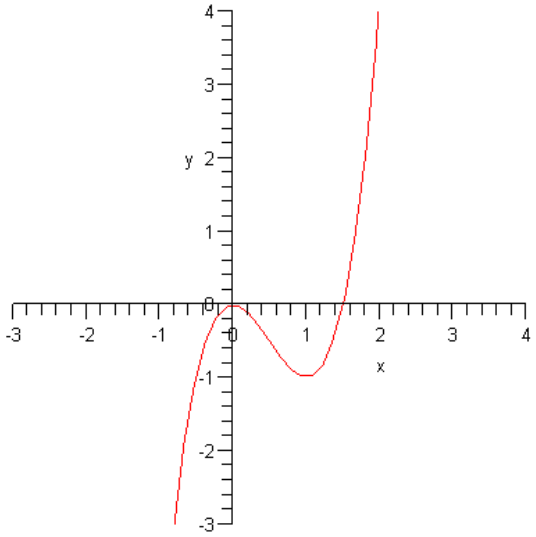
Testing the derivative

$f'(-1) = 6(-1)^2 - 6(-1) = 6 + 6 = 12$

$f'\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) = \frac{3}{2} - 3 = -\frac{3}{2}$

$f'(2) = 6(2)^2 - 6(2) = 24 - 12 = 12$

Interval	$(-\infty,0)$	$(0,1)$	$(1,\infty)$
Test Value	$x = -1$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	Positive	Negative	Positive
Conclusion	Increasing	Decreasing	Increasing



The function has a relative maximum at x = 0 and a relative minimum at $(0,0)$ and $(1,-1)$
(See diagram above)

Example 4

Find all extrema points of the function. $f(x) = -x^2$

$f(x) = -x^2$

$f'(x) = -2x$

$-2x = 0$

$\frac{-2x}{-2} = \frac{0}{-2}$

$x = 0$

$f'(-1) = -2(-1) = 2$

$f'(1) = -2(1) = -2$

Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	$x = -1$	$x = 1$
Sign of $f'(x)$	Positive	Negative
Conclusion	Increasing	Decreasing

The function has an absolute maximum at $(0,0)$

See Graph