Math 151
Section 2.7
Related Rates

Related Rates

Applications: When water is drained out of a conical tank, the variables for Volume, Radius, and Height of the water level are all functions of time.

$$V = \frac{\pi}{3} r^2 h$$

If you differentiate the above equations you the following related rates equation.

$$\frac{d}{dt}V = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$
$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2\right)' h + \frac{\pi}{3}(h)r^2$$
$$\frac{dV}{dt} = \frac{\pi}{3}(2rh)\frac{dr}{dt} + \frac{\pi}{3} r^2 \frac{dh}{dt}$$

Example 1

Assume that x and y are both differentiable functions of t and find the requires values of $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

Equation	**Find**	**Given**
1) $xy = 3$	$\frac{dy}{dt}$ *when* $x = 1$	$\frac{dx}{dt} = 2$
2) $x^2 + y^2 = 16$	$\frac{dy}{dt}$ *when* $x = 2, y = 2$	$\frac{dx}{dt} = 2$

1) First, find the derivative of $xy = 3$ implicitly

$$xy = 3$$

$$\frac{d}{dt}(xy) = \frac{d}{dt}(3)$$

$$\frac{d}{dt}(x)y + \frac{d}{dt}(y)x = 0$$

$$\frac{dx}{dt}y + \frac{dy}{dt}x = 0$$

Solve $xy = 3$ for x = 1

$$xy = 3 \Rightarrow (1)y = 3 \Rightarrow y = 3$$

Now, find $\frac{dy}{dt}$ when x = 1, y = 3 and $\frac{dx}{dt} = 2$

$$\frac{dx}{dt}y + \frac{dy}{dt}x = 0$$

$$(2)(3) + \frac{dy}{dt}(1) = 0$$

$$6 + \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -6$$

2) First, find the derivative of $x^2 + y^2 = 16$ implicitly

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(16)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Now, find $\frac{dy}{dt}$ when x = 2, y = 2 and $\frac{dx}{dt} = 2$

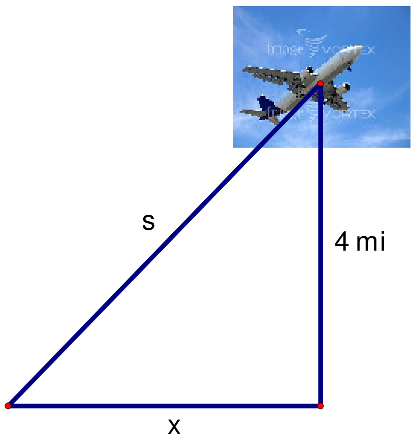
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(2)(2) + 2(2)\frac{dy}{dt} = 0$$

$$8 + 4\frac{dy}{dt} = 0 \Rightarrow 4\frac{dy}{dt} = -8 \Rightarrow \frac{dy}{dt} = -2$$

Example 2

An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown if figure 1.1 If s is decreasing at a rate of 440 miles per hour when s = 5 miles, what is the speed of the plane?



$$5^2 = x^2 + 3^2$$
$$25 = x^2 + 9$$
$$16 = x^2$$
$$x = 4$$

Using the Pythagorean Theorem, we get the following equation.

$$x^2 + 3^2 = s^2$$
$$x^2 + 9 = s^2$$

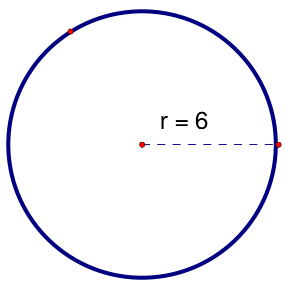
Differentiate the equation implicitly.

$$\frac{d}{dt}\left(x^2 + 9\right) = \frac{d}{dt}\left(s^2\right)$$
$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$
$$\frac{dx}{dt} = \frac{2s}{2x}\frac{ds}{dt}$$
$$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt} = \frac{5}{4}(-440) = -550\,mph$$

Example 3

The area of circle is increasing at a rate of 4 centimeters per minute. Find the rate of change of the area when a) r = 5 centimeters b) r = 12 centimeters

a) $\frac{dr}{dt} = 4$

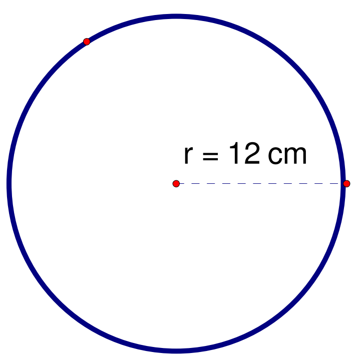


$$A = \pi r^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(5)(4) = 40\pi$$

b) $\frac{dr}{dt} = 4$



$$A = \pi r^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(12)(4) = 96\pi$$

Example 4

Find the rate of change of the volume of a cone, if $\frac{dr}{dt}$ is 2 inches per minutes and $h = 3r$ when $r = 8\ inches$. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$

$$h = 3r \Rightarrow \frac{dh}{dt} = 3 \cdot \frac{dr}{dt}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{d}{dt}V = \frac{d}{dt}\left(\pi r^2 h\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\frac{d}{dt}\left(r^2\right)h + \frac{\pi}{3}\frac{d}{dt}(h)r^2$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2r\frac{dr}{dt}h\right) + \frac{\pi}{3}\left(\frac{dh}{dt} \cdot r^2\right)$$

$$\frac{dV}{dt} = \frac{2\pi}{3}rh\frac{dr}{dt} + \frac{\pi}{3}r^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{2\pi}{3}r(3r)\frac{dr}{dt} + \frac{\pi}{3}\pi r^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = 2\pi r^2\frac{dr}{dt} + \frac{\pi}{3}r^2\left(3\frac{dr}{dt}\right)$$

$$\frac{dV}{dt} = 2\pi(8)^2(2) + \frac{\pi}{3}(3)^3(3(2))$$

$$\frac{dV}{dt} = 256\pi + 54\pi = 310\pi$$