

Section 2.6

Derivatives of Inverse Functions

Given the function $f(x) = x^3 + 4$, find the inverse.

$$f(x) = x^3 + 4$$

$$y = x^3 + 4$$

$$x = y^3 + 4$$

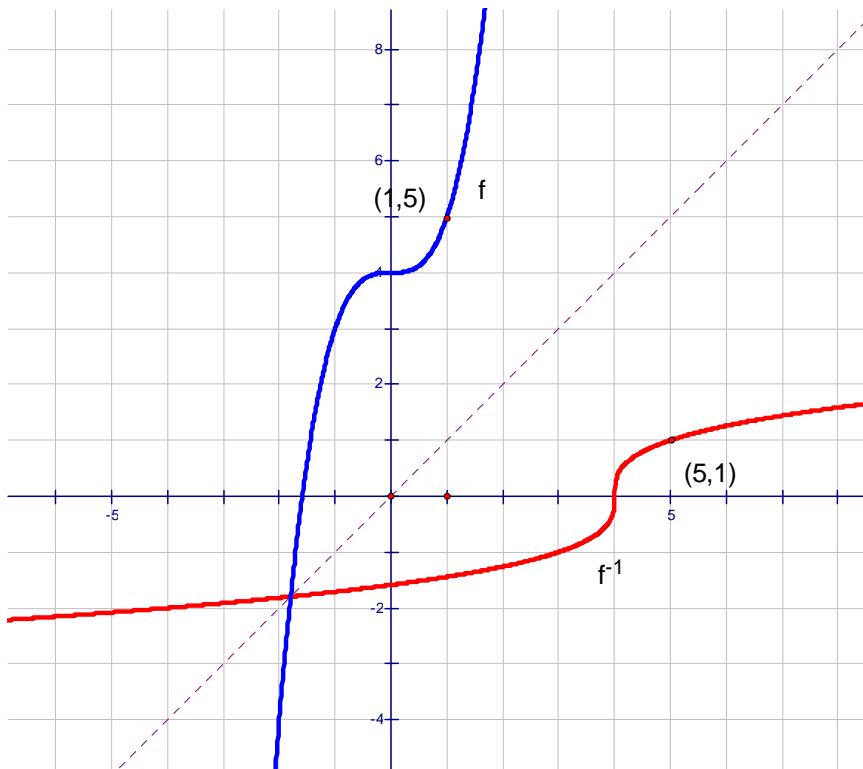
$$x - 4 = y^3 + 4 - 4$$

$$x - 4 = y^3$$

$$\sqrt[3]{x-4} = \sqrt[3]{y^3}$$

$$\sqrt[3]{x-4} = y \Rightarrow f^{-1}(x) = \sqrt[3]{x-4}$$

The graph of the function and its inverse are symmetric about $y = x$.



Now, look at the derivative of $f(x)$ at $(1,5)$ and the derivative of $f^{-1}(x)$ at $(5,1)$.

$$f(x) = x^3 + 4$$

$$f'(x) = 3x^2$$

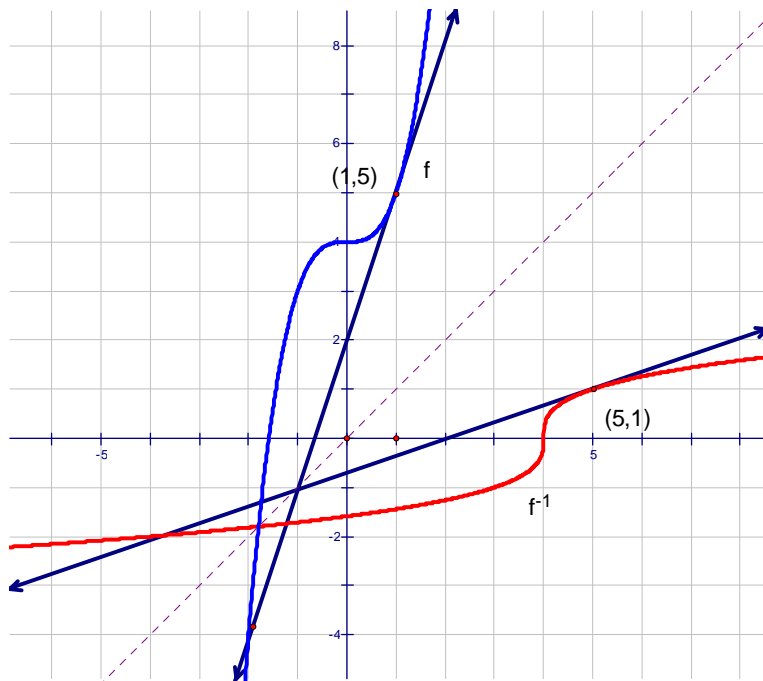
$$f'(1) = 3(1)^2 = 3(1) = 3$$

$$f(x) = (x-4)^{\frac{1}{3}} = u^{\frac{1}{3}} \text{ where } u = x-4 \Rightarrow \frac{du}{dx} = 1$$

$$(f^{-1}(x))' = \frac{1}{3}u^{\frac{1}{3}-1} \frac{du}{dx} = \frac{1}{3}u^{-\frac{2}{3}} \frac{du}{dx} = \frac{1}{3}(x-4)^{-\frac{2}{3}}(1) = \frac{1}{3(x-4)^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{(x-4)^2}}$$

$$(f^{-1}(5))' = \frac{1}{3\sqrt[3]{(5-4)^2}} = \frac{1}{3\sqrt[3]{1^2}} = \frac{1}{3\sqrt[3]{1}} = \frac{1}{3}$$

Note that the derivatives are reciprocals of each other.

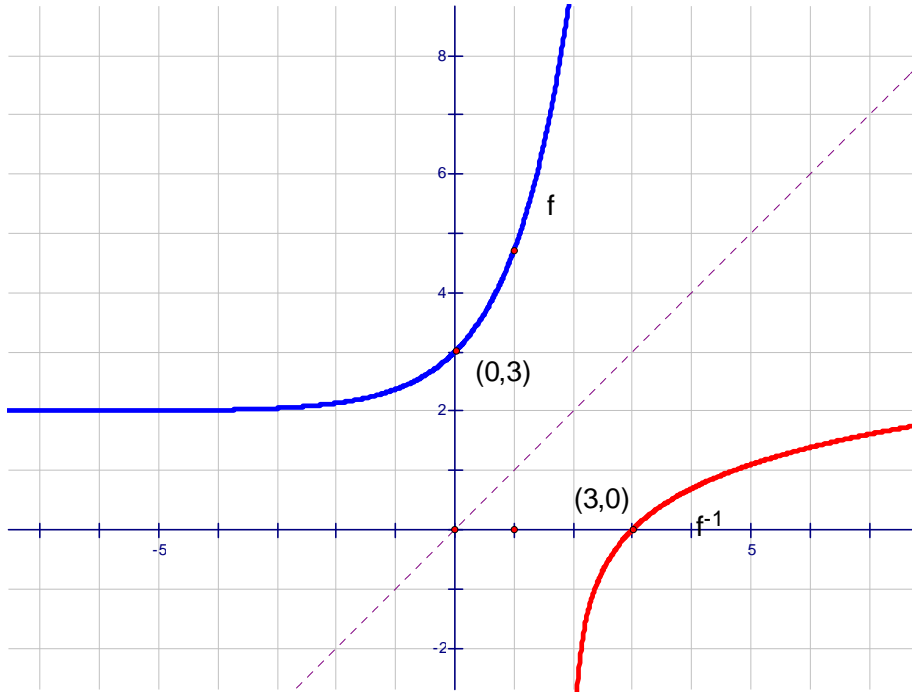


Example 2

Show that slopes of the graphs of $f(x)$ and $f^{-1}(x)$ are reciprocals at the indicated points

$$f(x) = e^x + 2 : (0,3)$$

$$f^{-1}(x) = \ln(x-2) : (3,0)$$



Compare their derivatives at the given points.

$$f(x) = e^x + 2$$

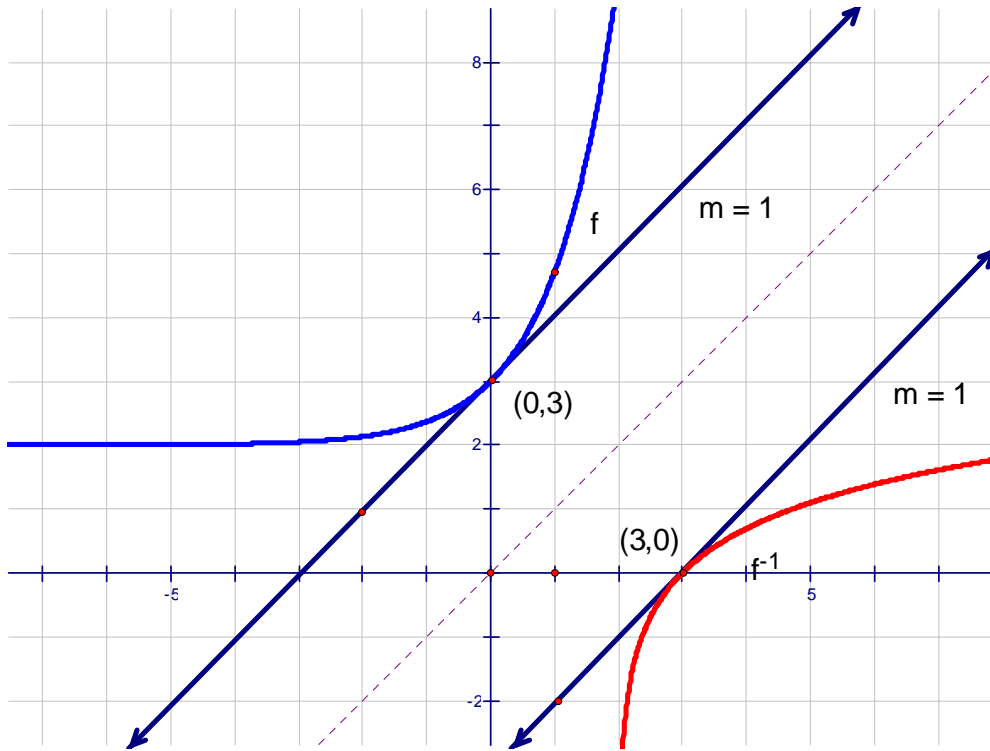
$$f'(x) = e^x$$

$$f'(0) = e^0 = 1$$

$$f^{-1}(x) = \ln(x-2)$$

$$(f^{-1}(x))' = \frac{1}{x-2}$$

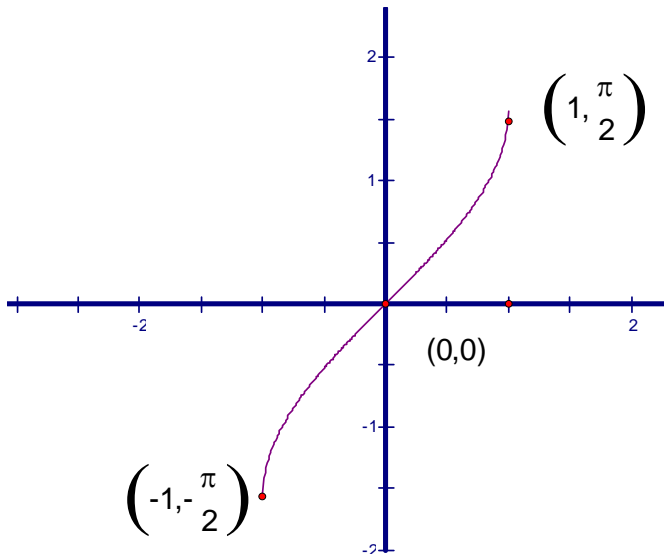
$$(f^{-1}(3))' = \frac{1}{3-2} = \frac{1}{1} = 1$$



The inverse of $\sin(x)$ is $\arcsin(x)$

The graph of $y = \arcsin(x)$

x	$y = \arcsin(x)$
-1	$y = \arcsin(-1) = -\frac{\pi}{2}$
$-\frac{\sqrt{2}}{2}$	$y = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$
0	$y = \arcsin(0) = 0$
$\frac{\sqrt{2}}{2}$	$y = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$
1	$y = \arcsin(1) = \frac{\pi}{2}$

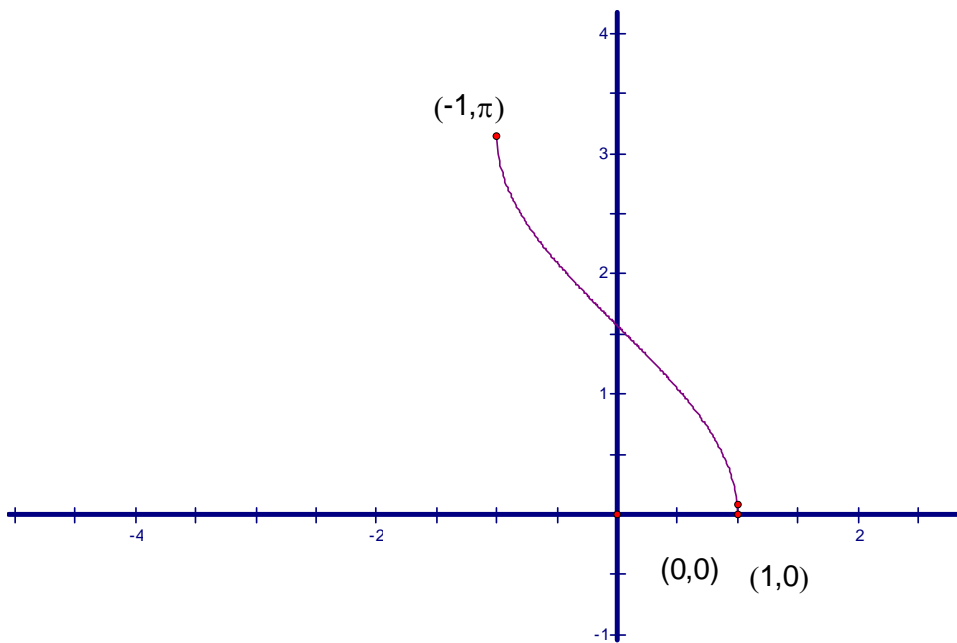


The derivative of $y = \arcsin(u)$

$$\frac{d}{dx}(\arcsin u) = \frac{u'}{\sqrt{1-u^2}}$$

The inverse function of $\cos(x)$ is $\arccos(x)$

The graph of $y = \arccos(x)$



The derivative of $y = \arccos(u)$

$$\frac{d}{dx}(\arccos u) = -\frac{u'}{\sqrt{1-u^2}}$$

The derivatives of the inverse trigonometric functions

$$\frac{d}{dx}(\arcsin u) = \frac{u'}{\sqrt{1-u^2}} \quad \text{(Proof see example 7)}$$

$$\frac{d}{dx}(\arccos u) = -\frac{u'}{\sqrt{1-u^2}} \quad \text{(Proof see example 8)}$$

$$\frac{d}{dx}(\arctan u) = \frac{u'}{1+u^2} \quad \text{(Proof see example 9)}$$

$$\frac{d}{dx}(\operatorname{arc cot} u) = -\frac{u'}{1+u^2}$$

$$\frac{d}{dx}(\operatorname{arc sec} u) = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}(\operatorname{arc csc} u) = -\frac{u'}{|u|\sqrt{u^2-1}}$$

Example 2

Find the derivative of $y = \arccos(4x)$

$$y = \arccos(4x)$$

$$y' = -\frac{(4x)'}{\sqrt{1-(4x)^2}} = -\frac{4}{\sqrt{1-16x^2}}$$

Example 3

Find the derivative of $y = \arccos(4x)$

$$y = \arcsin(3x^2)$$

$$y' = \frac{(3x^2)'}{\sqrt{1-(3x^2)^2}} = \frac{9x}{\sqrt{1-9x^4}}$$

Example 4

Find the derivative of $y = x^2 \arcsin(2x)$

$$y = x^2 \arcsin(2x)$$

$$y' = (x^2)' \arcsin(2x) + (\arcsin(2x))'(x^2)$$

$$y' = 2x \arcsin(2x) + \frac{(2x)'}{\sqrt{1-(2x)^2}}(x^2)$$

$$y' = 2x \arcsin(2x) + \frac{2x^2}{\sqrt{1-4x^2}}$$

Example 5

Find the derivative of $y = x^3 \arctan(5x)$

$$y = x^3 \arctan(5x)$$

$$y' = (x^3)' \arctan(5x) + (\arctan(5x))'(x^3) = 3x^2 \arctan(5x) + \frac{(5x)'}{1+(5x)^2}(x^3) = 3x^2 \arctan(5x) + \frac{5x^3}{1+25x^2}$$

Example 6

Find the derivative of $y = e^x \arcsin(x)$

$$y = e^x \arccos(x)$$

$$y' = (e^x) \arccos(x) + (\arccos(x))' (e^x) = e^x \arccos(x) + \frac{e^x}{\sqrt{1-x^2}}$$

Example 7**Proof of the derivative of the inverse sine function**

Let $y = \arcsin(u) = \sin^{-1}(u)$

By definition: $y = \sin^{-1}(u) \Leftrightarrow u = \sin y$

Note:

$$\sin^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \sin^2 y$$

$$\sqrt{\cos^2 y} = \sqrt{1 - \sin^2 y}$$

$$\cos y = \sqrt{1 - \sin^2 y}$$

Now use implicit differentiation:

$$u = \sin y$$

$$\frac{d}{dx} u = \frac{d}{dx} (\sin y)$$

$$u' = \cos y y'$$

$$\frac{u'}{\cos y} = \frac{\cos y y'}{\cos y}$$

$$y' = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1 - \sin^2 y}} = \frac{u'}{\sqrt{1 - u^2}}$$

Example 8

Proof of the derivative of the inverse cosine function

$$\text{Let } y = \arccos(u) = \cos^{-1}(u)$$

$$\text{By definition: } y = \cos^{-1}(u) \Leftrightarrow u = \cos y$$

Note:

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y = 1 - \cos^2 y$$

$$\sqrt{\sin^2 y} = \sqrt{1 - \cos^2 y}$$

$$\sin y = \sqrt{1 - \cos^2 y}$$

Now use implicit differentiation:

$$u = \cos y$$

$$\frac{d}{dx} u = \frac{d}{dx} (\cos y)$$

$$u' = -\sin y y'$$

$$\frac{u'}{-\sin y} = \frac{-\sin y y'}{-\sin y}$$

$$y' = -\frac{u'}{\sin y}$$

$$y' = -\frac{u'}{\sqrt{1 - \cos^2 y}}$$

$$y' = -\frac{u'}{\sqrt{1 - u^2}}$$

Example 9

Proof of the derivative of $\arctan(u)$

Let $y = \arctan(u) \Rightarrow y' = \tan^{-1}(u) \Leftrightarrow u = \tan y$

$$u = \tan y$$

$$\frac{d}{dx} u = \frac{d}{dx} \tan y$$

$$u' = \sec^2 y \cdot y'$$

$$y' = \frac{u'}{\sec^2 y}$$

$$y' = \frac{u'}{1 + \tan^2 y}$$

$$y' = \frac{u'}{1 + u^2}$$
