

Math 151
Section 2.4

Chain Rule

The chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 1

$$y = (3x^2 + 5)^5$$

Let $y = u^5$ where $u = 3x^2 + 5 \Rightarrow du = 6x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y' = 5u^4 du$$

$$y' = 5(3x^2 + 5x)^4 (6x)$$

$$y' = (30x)(3x^2 + 5x)^4$$

Example 2

$$y = (x^2 + 3x)^7$$

Let $y = u^7$ where $u = x^2 + 3x \Rightarrow du = 2x + 3$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y' = 7u^{7-1} du$$

$$y' = 7u^6 du$$

$$y' = 7(x^2 + 3x)^6 (2x + 3)$$

$$y' = (14x + 21)(x^2 + 3x)^6$$

Example 3

Find the derivative of $y = \sqrt{x^2 - 6x}$

$$y = \sqrt{x^2 - 6x}$$

$$y = (x^2 - 6x)^{\frac{1}{2}}$$

Let $y = u^{\frac{1}{2}}$ where $u = x^2 - 6x \Rightarrow du = 2x - 6$

$$y' = \frac{1}{2} u^{\frac{1}{2}-1} du$$

$$y' = \frac{1}{2} u^{-\frac{1}{2}} du$$

$$y' = \frac{1}{2} (x^2 - 6x)^{-\frac{1}{2}} (2x - 6)$$

$$y' = \frac{2x - 6}{2(x^2 - 6x)^{\frac{1}{2}}}$$

$$y' = \frac{2x - 6}{\sqrt{x^2 - 6x}}$$

Example 4

Find the derivative of $f(x) = \sqrt{x^3 - 3x}$

$$y = \sqrt{x^3 - 3x}$$

$$y = (x^3 - 3x)^{\frac{1}{2}}$$

Let $y = u^{\frac{1}{2}}$ where $u = x^3 - 3x \Rightarrow du = 3x^2 - 3$

$$y' = \frac{1}{2} u^{\frac{1}{2}-1} du = \frac{1}{2} u^{-\frac{1}{2}} du = \frac{1}{2} (x^3 - 3x)^{-\frac{1}{2}} (3x^2 - 3) = \frac{3x^2 - 3}{2(x^3 - 3x)^{\frac{1}{2}}} = \frac{3x^2 - 3}{2\sqrt{x^3 - 3x}}$$

Example 5

$$y = \sqrt[3]{x^2 + 4}$$

$$y = (x^2 + 4)^{\frac{1}{3}}$$

Let $y = u^{\frac{1}{3}}$ where $u = x^2 + 4 \Rightarrow u = 2x$

$$y' = \frac{1}{3} u^{\frac{1}{3}-1} du$$

$$y' = \frac{1}{3} u^{-\frac{2}{3}} du$$

$$y' = \frac{du}{3u^{\frac{2}{3}}}$$

$$y' = \frac{2x}{3(x^2 + 4)^{\frac{2}{3}}}$$

$$y' = \frac{2x}{3\sqrt[3]{x^2 + 4}}$$

Example 6

Find the derivative of $y = \cos(5x)$

$$y = \cos(5x)$$

$$u = 5x$$

$$du = 5$$

$$y = \cos u$$

$$y' = \cos u \cdot du$$

$$y' = 5 \cos(5x)$$

Example 7

Find the $y = e^{3x^2}$

$$y = e^{3x^2}$$

$$u = 3x^2$$

$$du = 6x$$

$$y' = e^u du$$

$$y' = e^{3x^2} (6x)$$

$$y' = 6xe^{3x^2}$$

Example 8

Find the $y = e^{x^2-6x}$

$$y = e^{x^2-6x}$$

$$u = x^2 - 6x$$

$$du = 2x - 6$$

$$y' = e^u du$$

$$y' = e^{x^2-6x} (2x - 6)$$

$$y' = (2x - 6)e^{x^2-6x}$$

Example 9

Find the derivative of $y = \cos(5x)$

$$y = e^{3x} \cos(6x)$$

$$y' = \frac{d}{dx} e^{3x} \cos(6x) + \frac{d}{dx} (\cos(6x)) e^{3x}$$

$$y' = 3e^{3x} \cos(6x) - 6e^{3x} \sin(6x)$$

$$y' = 3e^{3x} (\cos(6x) - 2 \sin(6x))$$

Example 10

Find the derivative of $y = 4xe^{x^2}$

$$y = 4xe^{x^2}$$

$$y' = \frac{d}{dx} (4x)e^{x^2} + \frac{d}{dx} e^{x^2} (4x)$$

$$y' = 4e^{x^2} + (4x)(2x)e^{x^2}$$

$$y' = 4e^{x^2} + 8x^2 e^{x^2}$$

$$y' = (4 + 8x^2)e^{x^2}$$

The derivative of the natural logarithm function

$$\frac{d}{dx} (\ln x) = \frac{1}{x}$$

Example 11

Find the derivative of $f(x) = 4 \ln(x)$

$$f'(x) = 4 \left(\frac{1}{x} \right) = \frac{4}{x}$$

Chain Rule for the natural logarithm

$$\frac{d}{dx}(\ln u) = \frac{1}{u} du = \frac{du}{u}$$

Example 12

Find the derivative of $f(x) = \ln(3x^3)$

Let $f(x) = \ln u$ where $u = 3x^3 \Rightarrow du = 9x^2$

$$f'(x) = \frac{du}{u} = \frac{9x^2}{3x^3} = \frac{3}{x}$$

Example 13

Find the derivative of $f(x) = e^{4x^2} \sin(3x)$

$$f'(x) = (e^{4x^2})'(\sin(3x)) + (\sin(3x))'(e^{4x^2})$$

$$f'(x) = 8xe^{4x^2} \sin(3x) + (3\cos(3x))e^{4x^2}$$

$$f'(x) = 8xe^{4x^2} \sin(3x) + 3e^{4x^2} \cos(3x)$$

Example 14

Find the derivative of $y = x^3 \ln(5x)$

$$y' = (x^3)' \ln(5x) + (\ln(5x))'(x^3)$$

$$y' = 3x^2 \ln(5x) + \frac{5}{5x} \cdot x^3$$

$$y' = 3x^2 \ln(5x) + x^2$$

Example 15

Find the slope of tangent to the function $y = (x^2 + 2x)^3$ at the point (1,27)

$$y = (x^2 + 2x)^3$$

$$\text{Let } y = u^3 \text{ where } u = x^2 + 2x \Rightarrow du = 2x + 2$$

$$y' = 3u^{3-1} du$$

$$y' = 3u^2 du$$

$$y' = 3(x^2 + 2x)^2 (2x + 2)$$

$$y' = (6x + 6)(x^2 + 2x)^2$$

Slope

$$y' = (6(1) + 6)(1^2 + 2(1))^2 = (6 + 6)(1 + 2)^2 = 12(3)^2 = 12(9) = 108$$

$$m = 108$$

Example 16

The equation of motion of a particle is $s(t) = t^3 - 6t$ where s is in meters and t is seconds.

a) Find the velocity and acceleration as a function of t.

$$v(t) = s'(t) = 3t^{3-1} - 6 = 3t^2 - 6$$

$$a(t) = s''(t) = 6t$$

b) Find the velocity after 2 seconds.

$$v(t) = 3t^{3-1} - 6 = 3t^2 - 6$$

$$v(2) = 3(2)^2 - 6 = 12 - 6 = 6 \frac{m}{s}$$

c) Find the acceleration after 2 seconds.

$$a(t) = s''(t) = 6t$$

$$a(2) = 6(2) = 12 \frac{m}{s^2}$$

Example 17

The equation of motion of a particle is $s(t) = 2t^3 - 12t$ where s is in meters and t is seconds.

- a) Find the velocity and acceleration as a function of t .

$$v(t) = s'(t) = 3 \cdot 2t^{3-1} - 12 = 6t^2 - 12$$

$$a(t) = s''(t) = 12t$$

- b) Find the velocity after 3 seconds.

$$v(t) = 6t^2 - 12$$

$$v(3) = 6(3)^2 - 12 = 54 - 12 = 42 \frac{m}{s}$$

- c) Find the acceleration after 3 seconds.

$$a(t) = s''(t) = 12t$$

$$a(3) = 12(3) = 36 \frac{m}{s^2}$$

Differentiability

A function is **differentiable** at a value x , if its derivative exists at x .

A function is **not differentiable** at a value x , if its derivative does not exist at x .

A function is not differentiable in the following situations.

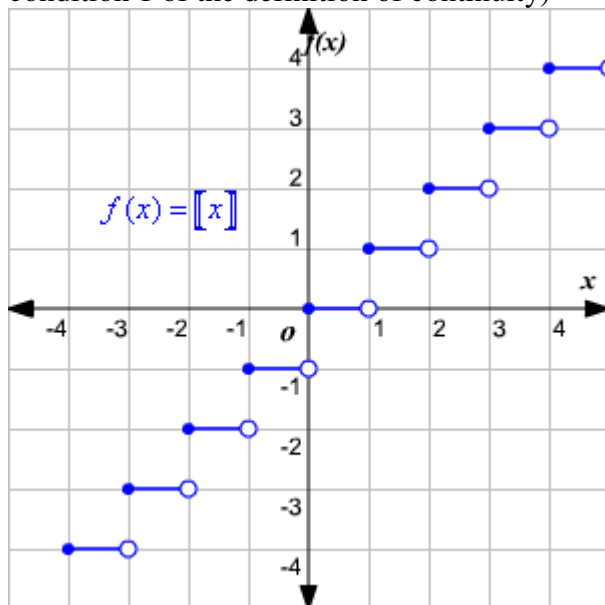
- 1) A value where the limit doesn't exist.

The function below does not have a limit at $x = 1$

$$\lim_{x \rightarrow 1^+} f(x) = 1 \text{ and } \lim_{x \rightarrow 1^-} f(x) = 0$$

Therefore, the limit does not exist because the left hand limit and right hand limit are different.

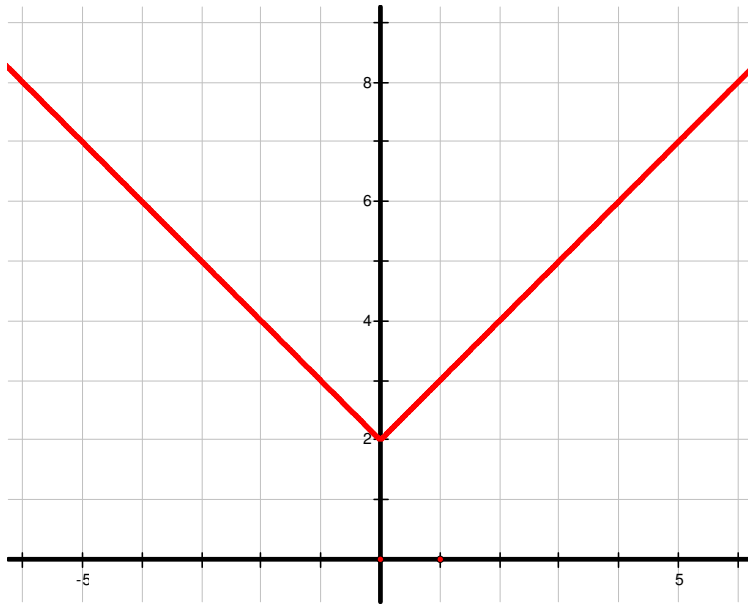
Since the limit doesn't exist at $x = 1$, the function is discontinuous at $x = 1$. (See condition 1 of the definition of continuity)



Graph courtesy http://hotmath.com/hotmath_help/topics/step-function.html

2) At a sharp edge in the graph of the function.

$$f(x) = |x| + 2 \text{ at } x = 0$$



This function is not differentiable at $x = 0$, because the derivative is different on both sides of $x = 0$

Right Hand Derivative:

$$f(x) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{|x| + 2 - (|0| + 2)}{x} = \lim_{x \rightarrow 0^+} \frac{|x| + 2 - 2}{x} = \lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$$

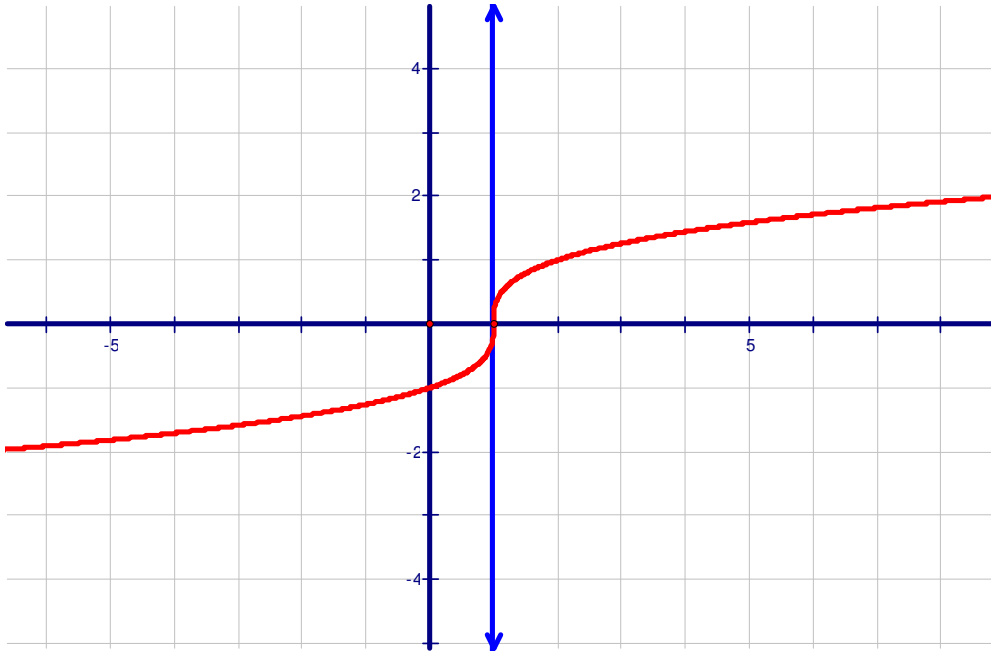
Left Hand Derivative:

$$f(x) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{|x| + 2 - (|0| + 2)}{x} = \lim_{x \rightarrow 0^-} \frac{|x| + 2 - 2}{x} = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$$

If you look at the slope of the tangent on the right hand side it is 1, where as the slope of the tangent line is -1 on the left hand side. Therefore, the derivative does not exist at $x = 0$

3) A vertical tangent line to the graph of the function.

$$f(x) = \sqrt[3]{x-1} \text{ at } x=1$$



The slope of the tangent line at $x=1$ is undefined because we have a vertical tangent line at the value $x=1$. The derivative is also undefined at $x=1$

$$f(x) = \sqrt[3]{x-1} \Rightarrow f(x) = (x-1)^{\frac{1}{3}}$$

$$u = x-1$$

$$\frac{du}{dx} = 1$$

$$f(x) = u^{\frac{1}{3}}$$

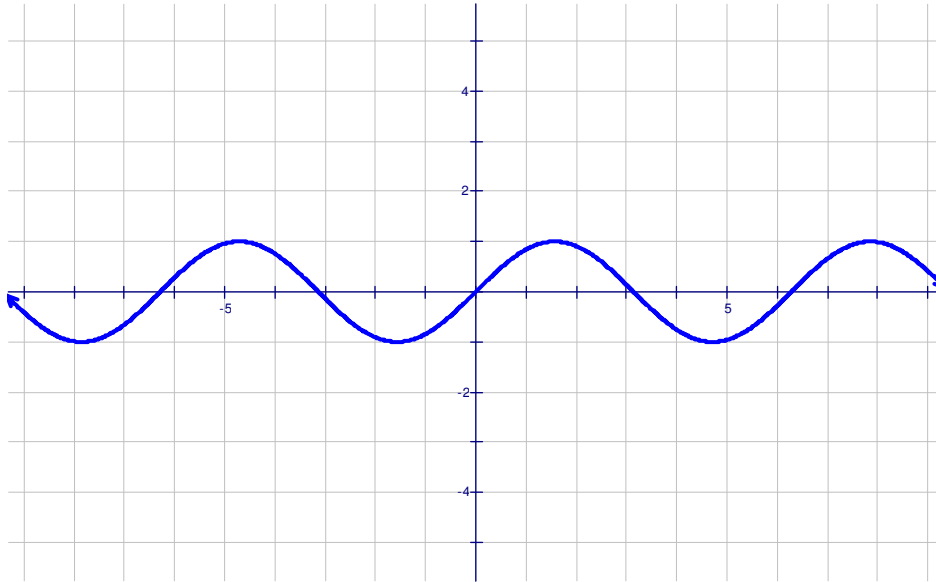
$$f(x) = \frac{1}{3} u^{-\frac{2}{3}} \frac{du}{dx} = \frac{1}{3} (x-1)^{-\frac{2}{3}} (1) = \frac{1}{3\sqrt[3]{(x-1)^2}}$$

$$f'(1) = \frac{1}{3\sqrt[3]{(1-1)^2}} = \frac{1}{3\sqrt[3]{0}} = \frac{1}{0}$$

Example 1

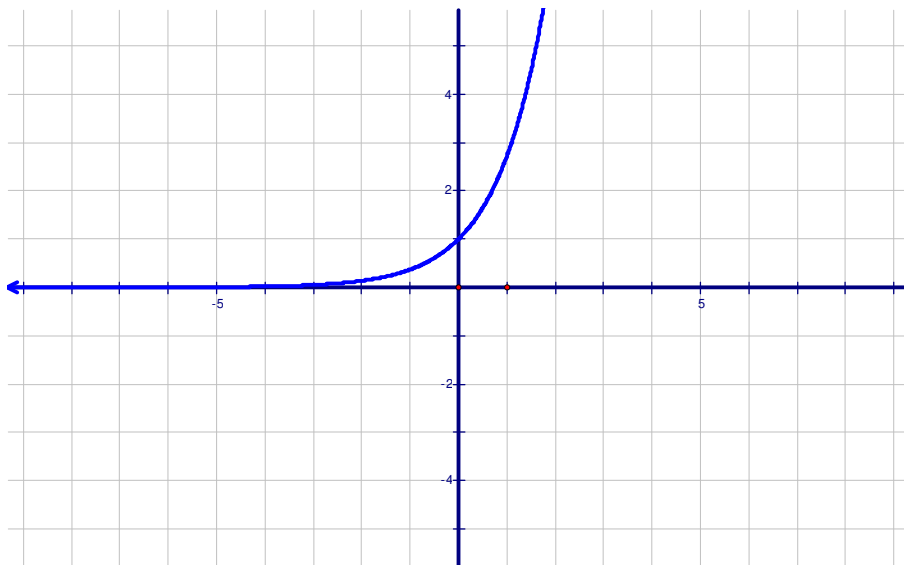
Use the graph of the function to determine what values of x the function is differentiable.

1) $f(x) = \sin x$



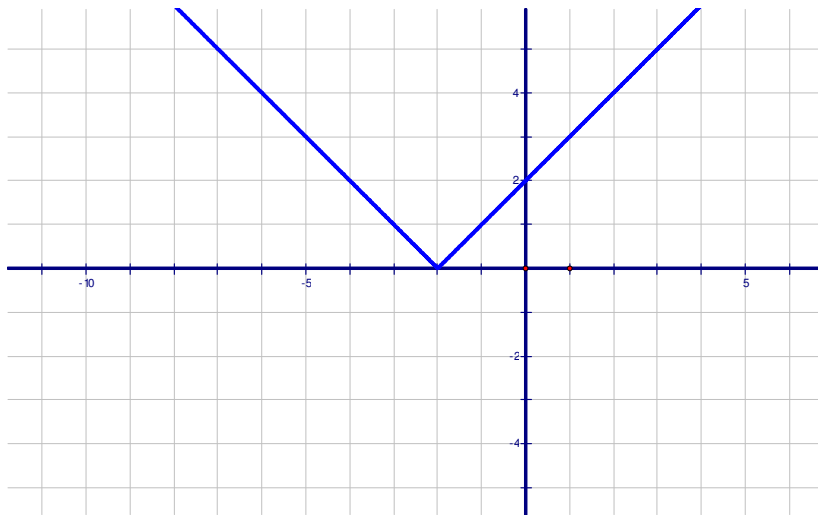
Since the graph has no sharp edges or vertical tangents the function is differentiable for all values of x .

2) $f(x) = e^{x+1}$



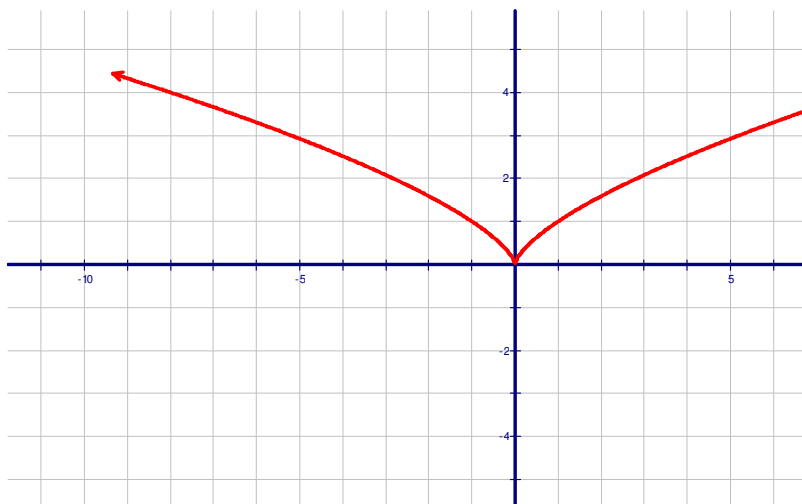
Since the graph has no sharp edges or vertical tangents the function is differentiable for all values of x .

3) $f(x) = |x + 2|$



This graph has a sharp edge at $x = -2$. Thus, the function is not differentiable at $x = -2$

4) $f(x) = x^{\frac{2}{3}}$



The graph has a vertical tangent line at $x = 0$. Also, note that the derivative does not exist at $x = 0$

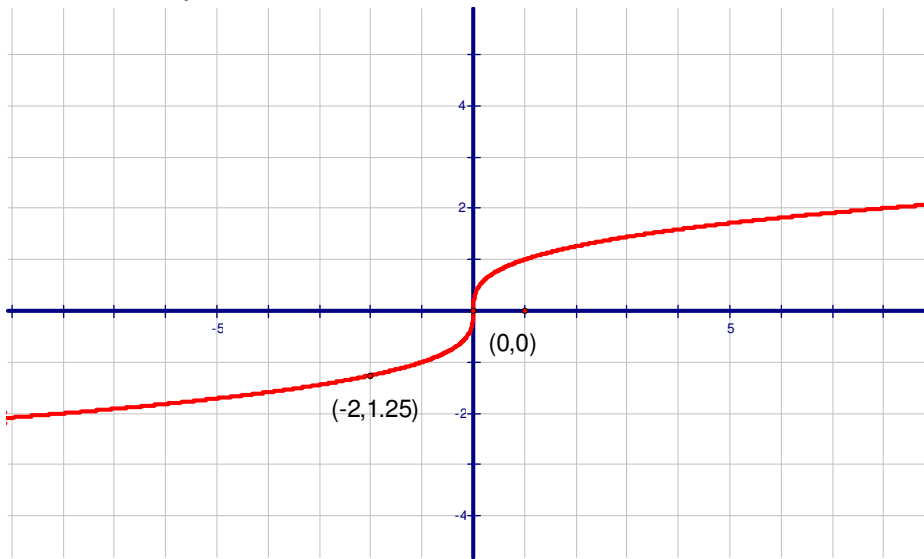
$$f(x) = x^{\frac{2}{3}}$$

$$\Rightarrow f'(x) = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{\sqrt[3]{x}}$$

$$\Rightarrow f'(0) = \frac{2}{\sqrt[3]{0}} = \frac{2}{0}$$

Example 2

Determine if $f(x) = \sqrt[3]{x}$ is differentiable at $x = -2$ and $x = 0$



The function is differentiable at $x = -2$

The function is not differentiable at $x = 0$ (The function has a vertical tangent at $x = 0$)

