

Math 151
Section 2.3

Product and Quotient Rule

The Product Rule

If $f(x) = F(x)S(x)$, then $f'(x) = F'(x)S(x) + S'(x)F(x)$

Quotient Rule

If $f(x) = \frac{T(x)}{B(x)}$, then $f'(x) = \frac{B(x)T'(x) - T(x)B'(x)}{(B(x))^2}$

Example 1

Given $f(x) = (x^2 - 4x)(x^2 - 3x)$, find $f'(x)$

$$f'(x) = \frac{d}{dx}(x^2 - 4x)(x^2 - 3x) + \frac{d}{dx}(x^2 - 3x)(x^2 - 4x)$$

$$f'(x) = (2x - 4)(x^2 - 3x) + (2x - 3)(x^2 - 4x)$$

$$f'(x) = 2x^3 - 6x^2 - 4x^2 - 12x + 2x^3 - 8x^2 - 3x^2 + 12x$$

$$f'(x) = 4x^3 - 21x^2 + 24x$$

Example 2

Given $f(x) = (5x^3 + 4x^2 + 4x)(2x^2 + 6x)$, find $f'(x)$

$$f'(x) = \frac{d}{dx}(5x^3 + 4x^2 + 4x)(2x^2 + 6x) + \frac{d}{dx}(2x^2 + 6x)(5x^3 + 4x^2 + 4x)$$

$$f'(x) = (15x^2 + 8x + 4)(2x^2 + 6x) + (4x + 6)(5x^3 + 4x^2 + 4x)$$

Example 3 Given $f(x) = 4x^3e^x$, find $f'(x)$

$$f(x) = \frac{d}{dx}(4x^3)e^x + \frac{d}{dx}(e^x)$$

$$f'(x) = 12x^2e^x + 4x^3e^x$$

$$\text{or } f'(x) = 4x^2e^x(3+x)$$

Example 4

Find the derivative of $f(x) = e^x \sin x$

$$f'(x) = \frac{d}{dx}(e^x)\sin x + \frac{d}{dx}(\sin x)e^x$$

$$f'(x) = e^x \sin x + e^x \cos x$$

Example 5

Given $f(x) = \frac{x^2}{3x^3 - 4x + 5}$, find $f'(x)$

$$f(x) = \frac{(3x^3 - 4x + 5)\frac{d}{dx}(e^x) - (x^2)\frac{d}{dx}(3x^3 - 4x + 5)}{(3x^3 - 4x + 5)^2}$$

$$f'(x) = \frac{(3x^3 - 4x + 5)(2x) - (x^2)(9x^2 - 4)}{(3x^3 - 4x + 5)^2}$$

$$f'(x) = \frac{6x^4 - 8x^2 + 10x - 9x^4 + 4x^2}{(3x^3 - 4x + 5)^2}$$

$$f(x) = \frac{-3x^4 - 4x^2 + 10x}{(3x^3 - 4x + 5)^2}$$

Example 6

Given $f(x) = \frac{x^2 - 5x}{x^2 - 3x + 3}$, then $f'(x)$

$$f'(x) = \frac{(x^2 - 3x + 3)(2x - 5) - (x^2 - 5x)(2x - 3)}{(x^2 - 3x + 3)^2}$$

$$f'(x) = \frac{2x^3 - 6x^2 + 6x - 5x^2 + 15x + 15 - (2x^3 - 10x^2 - 3x^2 + 15x)}{(x^2 - 3x + 3)^2}$$

$$f'(x) = \frac{2x^3 - 11x^2 + 21x + 15 - 2x^3 + 13x^2 - 15x}{(x^2 - 3x + 3)^2}$$

$$f'(x) = \frac{2x^2 + 6x + 15}{(x^2 - 3x + 3)^2}$$

Example 7

Given $f(x) = \frac{e^x}{x^2 - 2}$, then $f'(x)$

$$f'(x) = \frac{(x^2 - 2)\frac{d}{dx}(e^x) + (x^2 - 2)\frac{d}{dx}(e^x)}{(x^2 - 2)^2}$$

$$f'(x) = \frac{(x^2 - 2)e^x + e^x(2x)}{(x^2 - 2)^2}$$

$$f'(x) = \frac{x^2e^x - 2e^x + 2xe^x}{(x^2 - 2)^2}$$

$$f'(x) = \frac{(x^2 + 2x - 2)e^x}{(x^2 - 2)^2}$$

The derivatives of the other trigonometric functions

The derivative of tangent

Find the derivative of $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$

$$f(x) = \frac{(\cos(x))(\sin(x))' - (\sin(x))(\cos(x))'}{(\sin(x))^2}$$

$$f'(x) = \frac{\cos(x) \cdot \cos(x) - (\sin(x))(-\sin(x))}{\sin^2(x)}$$

$$f'(x) = \frac{\cos^2(x) + \sin^2(x)}{\sin^2(x)}$$

$$f'(x) = \frac{1}{\sin^2(x)}$$

$$f'(x) = \csc^2(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

The derivative of cosecant

$$f(x) = \csc(x) = \frac{1}{\sin(x)}$$

$$f'(x) = \frac{(\sin(x)) \cdot (1)' - 1 \cdot (\sin(x))'}{(\sin(x))^2}$$

$$f'(x) = \frac{\sin(x) \cdot 0 - 1(\cos(x))}{\sin^2(x)}$$

$$f'(x) = \frac{-\cos(x)}{\sin^2(x)}$$

$$f'(x) = -\cos(x) \cdot \frac{1}{\sin^2(x)}$$

$$f'(x) = -\cos(x) \csc^2(x)$$

$$\frac{d}{dx}(\csc(x)) = -\cos(x) \csc^2(x)$$

The derivatives of the other trigonometric functions

$$\frac{d}{dx}(\csc(x)) = -\csc(x) \cdot \cot(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x) \cdot \tan(x)$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Example 8

Find the derivative of $g(x) = \frac{\cos x}{x^3}$

$$\begin{aligned} g'(x) &= \frac{(x^3)(\cos x)' - (\cos x)(x^3)'}{(x^3)^2} \\ &= \frac{x^3(-\sin x) - (\cos x)(3x^2)}{x^6} \\ &= \frac{-x^2(x \sin x + 3 \cos x)}{x^6} \\ &= \frac{-(x \sin x + 3 \cos x)}{x^4} \end{aligned}$$

Example 9

Find the equation of a tangent line to the function $f(x) = x^3e^x$ at the point $(1, e)$

$$f(x) = \frac{d}{dx}(x^3)e^x + \frac{d}{dx}(e^x)(x^3)$$

$$f'(x) = 3x^2e^x + x^3e^x$$

$$f'(1) = 3(1)^2e^1 + (1)^3e^1 = 3e + 1e = 4e$$

$$y - e = 4e(x - 1)$$

$$y - e = 4ex - 4e$$

$$y = 4ex - 3e$$

$$y = e(4x - 3)$$

Example 10

Find the slope of a tangent line to the curve given by the function $f(x) = \frac{x^2}{x+3}$ through

the point $\left(-1, -\frac{1}{2}\right)$

First find the derivative of the function: $f(x) = \frac{x^2}{x+3}$

$$f'(x) = \frac{(x+3)\frac{d}{dx}(x^2) - (x^2)\frac{d}{dx}(x+3)}{(x+3)^2} = \frac{(x+3)(2x) - (x^2)(1)}{(x+3)^2} = \frac{2x^2 + 6x - x^2}{(x+3)^2} = \frac{x^2 + 6x}{(x+3)^2}$$

Find the value of the derivative at $x = -1$

$$f'(-1) = \frac{(-1) + 6(-1)}{(-1+3)^2} = \frac{-1-6}{2^2} = -\frac{7}{4}$$

Example 11

Find the derivative of $f(x) = x^4 \sin x$

$$f'(x) = (x^4)' \sin x + (\sin x)' (x^4)$$

$$f'(x) = 4x^3 \sin x + \cos x (x^4)$$

$$f'(x) = 4x^3 \sin x + x^4 \cos x$$
