

Math 151

Section 1.6

Evaluating Limits Analytically

Recall from example 1: $\lim_{x \rightarrow 1} x^2 + 1 = 2$

Definition 1: $\lim_{x \rightarrow c} f(x) = f(c)$

Thus the limit in the above example can be found by using definition 1,

$$\lim_{x \rightarrow 1} f(x) = f(1) = 1^2 + 1 = 1 + 1 = 2$$

Example 1

Find $\lim_{x \rightarrow 3} x^3$

Solution: $\lim_{x \rightarrow 3} x^3 = 3^3 = 27$

Example 2

Find $\lim_{x \rightarrow -4} x^2 + 5x$

Solution: $\lim_{x \rightarrow -4} x^2 + 5x = (-4)^2 + 5(-4) = 16 - 20 = -4$

Example 3

Find $\lim_{x \rightarrow 3} \frac{5}{x-3}$

Solution: $\lim_{x \rightarrow 3} \frac{5}{x-3} = \frac{5}{3-3} = \frac{5}{0}$ *Undefined*

Limit does not exist

Example 4

Find $\lim_{x \rightarrow 1} \sqrt{2x-5}$

Solution: $\lim_{x \rightarrow 1} \sqrt{2x-5} = \sqrt{2(1)-5} = \sqrt{2-5} = \sqrt{-3}$ Undefined

Limit does not exist

Example 5

Find $\lim_{x \rightarrow 3} \frac{x-3}{x^2-9}$

Solution: $\lim_{x \rightarrow 3} \frac{x-3}{x^2-9} = \lim_{x \rightarrow 3} \frac{x-3}{(x-3)(x+3)} = \lim_{x \rightarrow 3} \frac{1}{x+3} = \frac{1}{3+3} = \frac{1}{6}$

In example 5, if you substitute 3 in for x before you reduce the polynomial down, you will get an undetermined form. $\frac{0}{0}$

Example 6

Find $\lim_{x \rightarrow 2} \frac{2-x}{x^2-4}$

Solution: $\lim_{x \rightarrow 2} \frac{2-x}{x^2-4} = \lim_{x \rightarrow 3} \frac{-(x-2)}{(x-2)(x+2)} = \lim_{x \rightarrow 3} \frac{-1}{x+2} = \frac{-1}{2+2} = -\frac{1}{4}$

Example 7

Find $\lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t^2 - 1}$

Solution: $\lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t^2 - 1} = \lim_{t \rightarrow 1} \frac{(t+2)(t-1)}{(t-1)(t+1)} = \lim_{t \rightarrow 1} \frac{t+2}{t+1} = \frac{1+2}{1+1} = \frac{3}{2}$

Example 8

Find $\lim_{x \rightarrow -1} \frac{4x - 5}{3 - x}$

Solution: $\lim_{x \rightarrow -1} \frac{4x - 5}{3 - x} = \frac{4(-1) - 5}{3 - (-1)} = \frac{-4 - 5}{4} = -\frac{9}{4}$

Example 9

Find $\lim_{x \rightarrow 2} e^{x-2}$

Solution: $\lim_{x \rightarrow 2} e^{x-2} = e^{2-2} = e^0 = 1$

Example 10

Find $\lim_{x \rightarrow \frac{\pi}{2}} \sin x$

Solution: $\lim_{x \rightarrow \frac{\pi}{2}} \sin x = \sin\left(\frac{\pi}{2}\right) = 1$

Example 11

Find $\lim_{h \rightarrow 1} \tan \frac{\pi h}{4}$

Solution:

$$\lim_{h \rightarrow 1} \tan \frac{\pi h}{4} = \tan \frac{\pi(1)}{4} = \tan\left(\frac{\pi}{4}\right) = 1$$

Example 12

Find $\lim_{h \rightarrow 0} \frac{(h+1)^2 - 1}{h}$

Solution:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(h+1)^2 - 1}{h} &= \lim_{h \rightarrow 0} \frac{(h+1)(h+1) - 1}{h} = \lim_{h \rightarrow 0} \frac{h^2 + h + h + 1 - 1}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 2h}{h} = \lim_{h \rightarrow 0} \frac{h(h+2)}{h} \\ &= \lim_{h \rightarrow 0} h + 2 = 0 + 2 = 2 \end{aligned}$$

Example 13

Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \rightarrow 0} \frac{(\sqrt{t^2 + 9} - 3)(\sqrt{t^2 + 9} + 3)}{t^2(\sqrt{t^2 + 9} + 3)} = \lim_{t \rightarrow 0} \frac{(\sqrt{t^2 + 9})^2 + 3\sqrt{t^2 + 9} - 3\sqrt{t^2 + 9} - 9}{t^2(\sqrt{t^2 + 9} + 3)} \\ &= \lim_{t \rightarrow 0} \frac{t^2 + 9 - 9}{t^2(\sqrt{t^2 + 9} + 3)} = \lim_{t \rightarrow 0} \frac{t^2}{t^2(\sqrt{t^2 + 9} + 3)} = \lim_{t \rightarrow 0} \frac{1}{(\sqrt{t^2 + 9} + 3)} = \frac{1}{\sqrt{0^2 + 9} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3 + 3} = \frac{1}{6} \end{aligned}$$

Example 14

Find $\lim_{x \rightarrow 2} \frac{x-2}{x^3-8}$

$$\lim_{x \rightarrow 2} \frac{x-2}{x^3-8} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2x+2^2)} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x^2+2x+4)} = \lim_{x \rightarrow 2} \frac{1}{x^2+2x+4} = \frac{1}{2^2+2(2)+4} = \frac{1}{12}$$

Limit Laws

1) $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$

2) $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$

3) $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$

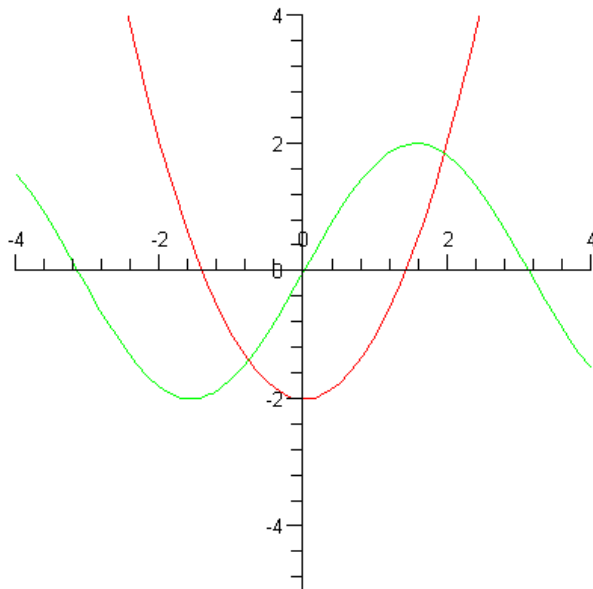
4) $\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$

5) $\lim_{x \rightarrow a} c \cdot f(x) = c \cdot \lim_{x \rightarrow a} f(x)$

Example 13

Find $\lim_{x \rightarrow 0} \frac{\sqrt{5-x} - \sqrt{5}}{x}$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{5-x} - \sqrt{5}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{5-x} - \sqrt{x}}{x} \cdot \frac{\sqrt{5-x} + \sqrt{5}}{\sqrt{5-x} + \sqrt{5}} = \lim_{x \rightarrow 0} \frac{(\sqrt{5-x})^2 - \sqrt{5}\sqrt{5-x} + \sqrt{5}\sqrt{5-x} - 5}{x(\sqrt{5-x} + \sqrt{5})} \\ &= \lim_{x \rightarrow 0} \frac{5-x-5}{x(\sqrt{5-x} + \sqrt{5})} = \lim_{x \rightarrow 0} \frac{-x}{x(\sqrt{5-x} + \sqrt{5})} = \lim_{x \rightarrow 0} \frac{-1}{\sqrt{5-x} + \sqrt{5}} = \frac{-1}{\sqrt{5-0} + \sqrt{5}} = \frac{-1}{2\sqrt{5}} \end{aligned}$$

Example 16

$f(x)$ – green line
 $g(x)$ – red line

Use to the above graph to evaluate each limit

1) Find $\lim_{x \rightarrow 0} (f(x) + g(x))$

$$\lim_{x \rightarrow 0} (f(x) + g(x)) = \lim_{x \rightarrow 0} f(x) + \lim_{x \rightarrow 0} g(x) = 0 - 2 = -2$$

2) Find $\lim_{x \rightarrow 0} (f(x) - g(x))$

$$\lim_{x \rightarrow 0} (f(x) - g(x)) = \lim_{x \rightarrow 0} f(x) - \lim_{x \rightarrow 0} g(x) = 0 - (-2) = 0 + 2 = 2$$

3) Find $\lim_{x \rightarrow 0} (f(x) \cdot g(x))$

$$\lim_{x \rightarrow 0} (f(x) \cdot g(x)) = \lim_{x \rightarrow 0} f(x) \cdot \lim_{x \rightarrow 0} g(x) = 0 \cdot (-2) = 0$$

4) $\lim_{x \rightarrow 0} \left(\frac{f(x)}{g(x)} \right) = \lim_{x \rightarrow 0} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow 0} f(x)}{\lim_{x \rightarrow 0} g(x)} = \frac{0}{-2} = 0$