Math 151
Section 1.3

Inverse Functions

Definition of an inverse function

A function $f^{-1}(x)$ is an inverse function of $f(x)$ if the following conditions are true.

$$f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x$$

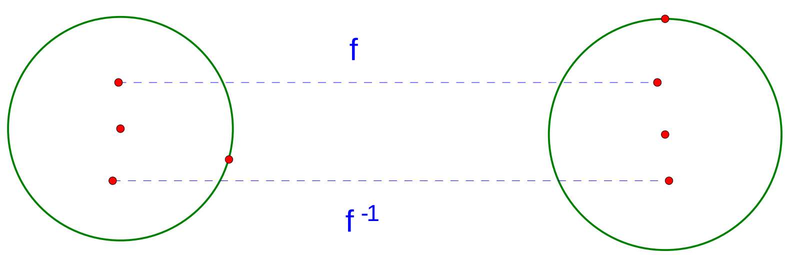
The **inverse function** is formed by interchanging the first and second coordinates.

Given the following function, find the inverse function.

$$f(x) = \{(1,2),(3,4),(4,2),(5,6)\}$$

Inverse Functions:

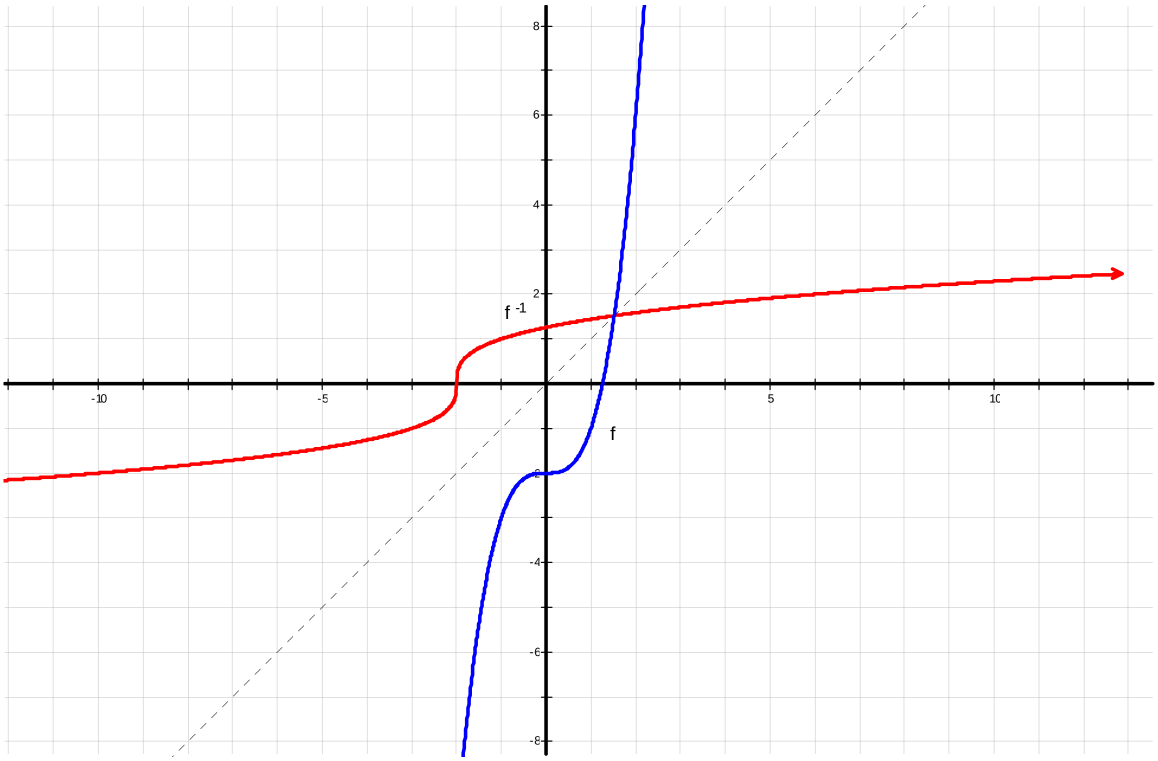
$$f^{-1}(x) = \{(2,1),(4,3),(2,4),(6,5)\}$$



Reflexive Property of Inverse Function

The graph of f contains the point (a,b) if and only if the graph of f^{-1} contains the point (b,a)

Graph of the function verses its inverse function.



The graph of a function and its inverse are symmetric about the line y = x.

Example 1

Show that f and g are inverses by showing that $f(g(x)) = x$ *and* $g(f(x)) = x$

$$f(x) = 2x + 4 : g(x) = \frac{x-4}{2}$$

Part 1: $f(g(x)) = f\left(\frac{x-4}{2}\right) = 2\left(\frac{x-4}{2}\right) + 4 = (x-4) + 4 = x$

Part 2: $g(f(x)) = g(2x+4) = \frac{(2x+4)-4}{2} = \frac{2x}{2} = x$

Example 2

Show that f and g are inverses by showing that $f(g(x)) = x \; and \; g(f(x)) = x$

$f(x) = x^3 + 1 : g(x) = \sqrt[3]{x-1}$

Part 1: $f(g(x)) = f\left(\sqrt[3]{x-1}\right) = \left(\sqrt[3]{x-1}\right)^3 + 1 = x - 1 + 1 = x$

Part 2: $g(f(x)) = g\left(x^3 + 1\right) = \sqrt[3]{\left(x^3 + 1\right) - 1} = \sqrt[3]{x^3} = x$

Example 3

Show that f and g are inverses by showing that $f(g(x)) = x \; and \; g(f(x)) = x$

$f(x) = x^2 - 2 : g(x) = \sqrt{x+2} : x \geq 0$

Part 1: $f(g(x)) = f\left(\sqrt{x+2}\right) = \left(\sqrt{x+2}\right)^2 - 2 = x + 2 - 2 = x$

Part 2: $g(f(x)) = g\left(x^2 - 2\right) = \sqrt{(x-2)+2} = \sqrt{x^2} = x$

Example 4

Find the inverse of $f(x) = x^3 - 4$

First, replace $f(x)$ with y

$y = x^3 - 4$

Next, invert x and y

$x = y^3 - 4$

Solve for y

$$x = y^3 - 4$$
$$x + 4 = y^3 - 4 + 4$$
$$x + 4 = y^3$$
$$\sqrt[3]{x+4} = \sqrt[3]{y^3}$$
$$\sqrt[3]{x+4} = y$$
$$\Rightarrow f^{-1}(x) = \sqrt[3]{x+4}$$

Example 5

Find the inverse of $f(x) = 4x + 3$

First, replace $f(x)$ with y

$$y = 4x + 3$$

Next, invert x and y

$$x = 4y + 3$$

Solve for y

$$x = 4y + 3$$
$$x - 3 = 4y + 3 - 3$$
$$x - 3 = 4y$$
$$\frac{x-3}{4} = \frac{4y}{4}$$
$$\frac{x-3}{4} = y \Rightarrow f^{-1}(x) = \frac{x-3}{4}$$

The existence of an inverse function

A function has an inverse if and only if it is a one-to-one function.

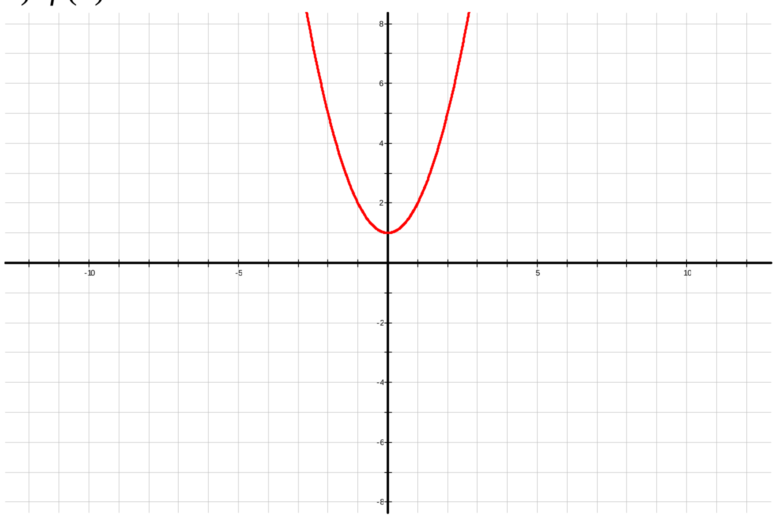
Horizontal Line Test

If a horizontal line can be drawn on a graph of a function so that it intercepts a given function more than once, the function is not one-to-one.

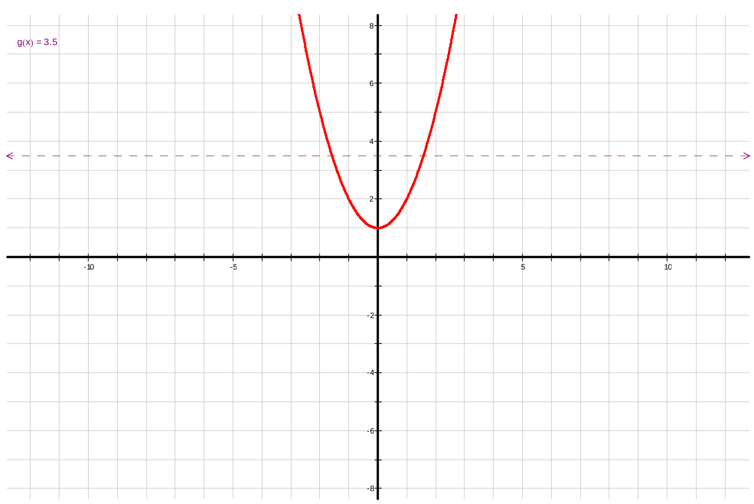
Example 6

Use the horizontal line test to determine if the function is one-to-one.

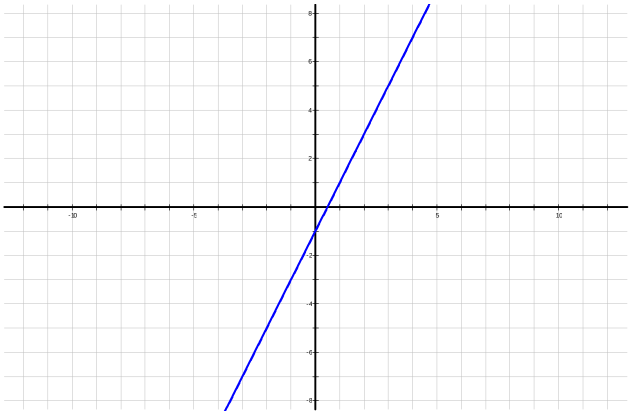
1) $f(x) = x^2 + 1$



Solution: The function fails the horizontal line test, so the function is not a one-to-one function.

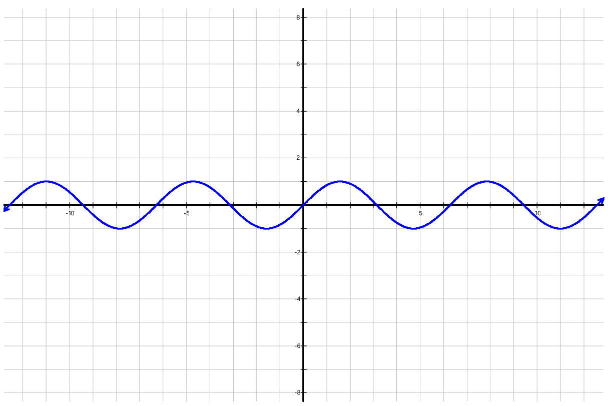


2) $f(x) = 2x - 1$

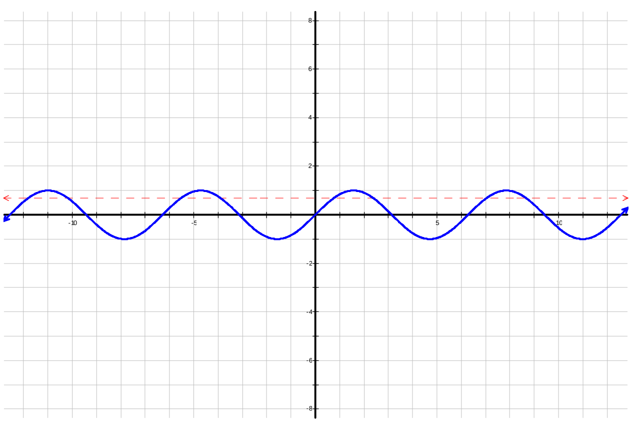


The function passes the horizontal line test, because every horizontal line intersects the function only once. Therefore, the function is one-to-one and has an inverse.

3) $f(x) = \cos(x)$



Solution: The function fails the horizontal line test, so the function is not a one-to-one function.



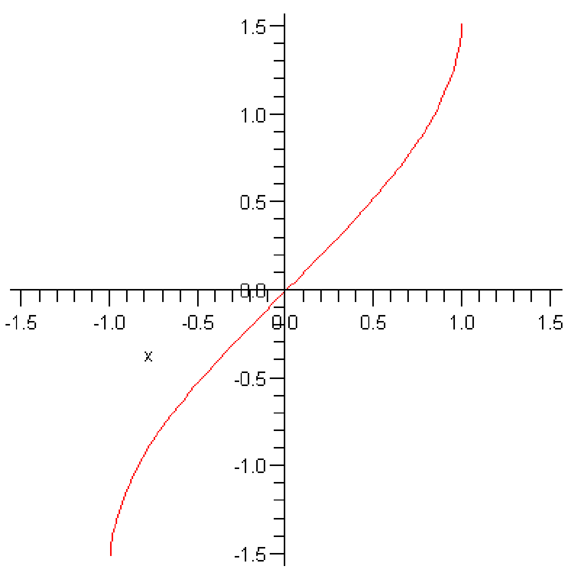
Inverse Trigonometric Functions

Function	**Domain**	**Range**
$y = \arcsin x \ \ iff \ \ \sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x \ \ iff \ \ \cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x \ \ iff \ \ \tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = arc\cot x \ \ iff \ \ \sin y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = arc\sec x \ \ iff \ \ \sec y = x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi : y \ne \frac{\pi}{2}$
$y = arc\csc x \ \ iff \ \ \csc y = x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2} : y \ne 0$

Graphs of the Inverse Functions

Example 7 (The Graph of $y = \arcsin x$)

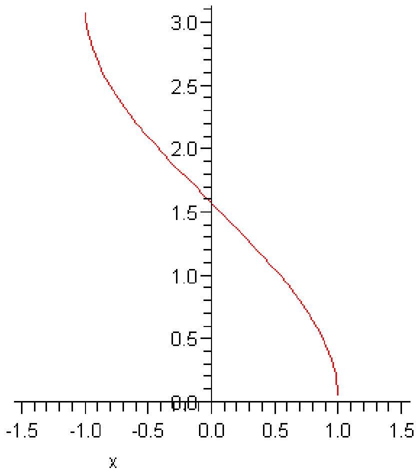
Domain: $-1 \le x \le 1 \ \ or \ \ [-1,1]$ **Range:** $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \ \ or \ \ \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Example 8

The Graph of $y = \arccos x$

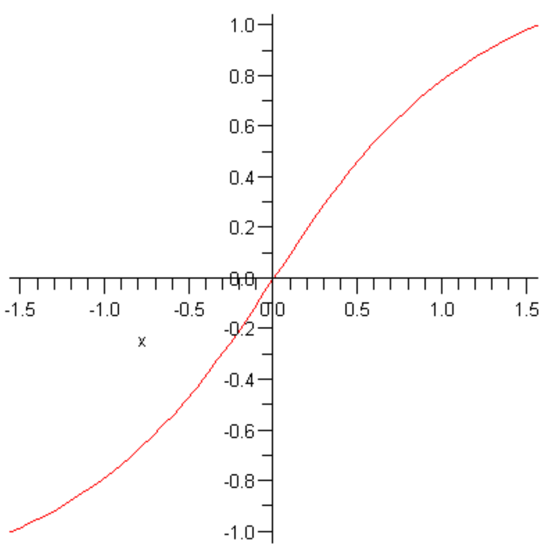
Domain: $-1 \le x \le 1$ *or* $[-1,1]$ **Range:** $0 \le y \le \pi$ *or* $[0, \pi]$



Example 9

The Graph of $y = \arctan x$

Domain: $-\infty < x < 1$ *or* $(-\infty, \infty)$ **Range:** $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ *or* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Example 10

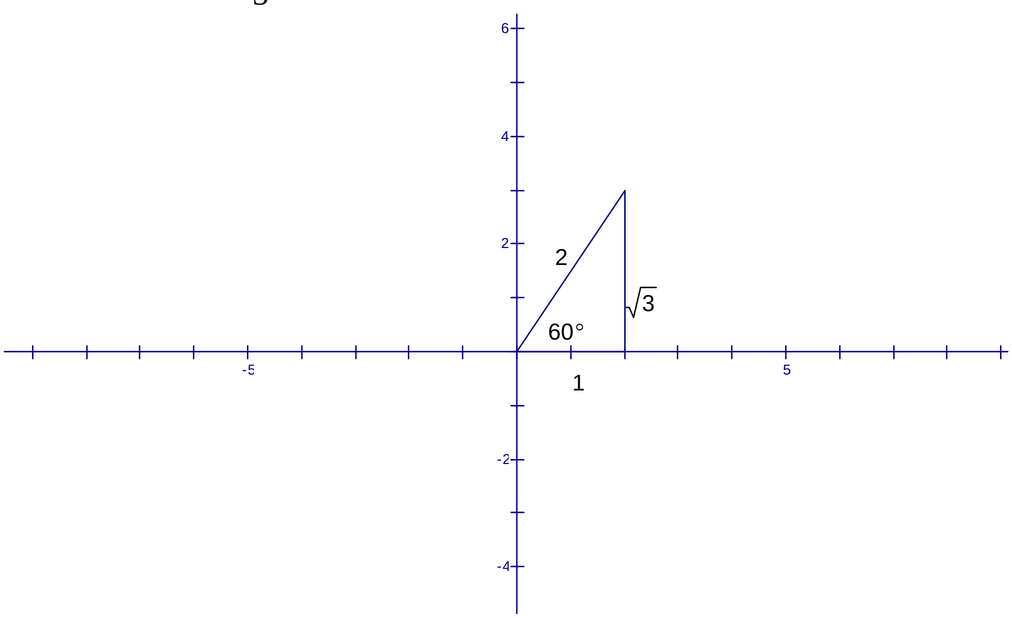
Evaluate each of the following inverse trig functions

1) Evaluate $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

$$\arcsin\left(\frac{\sqrt{3}}{2}\right)$$

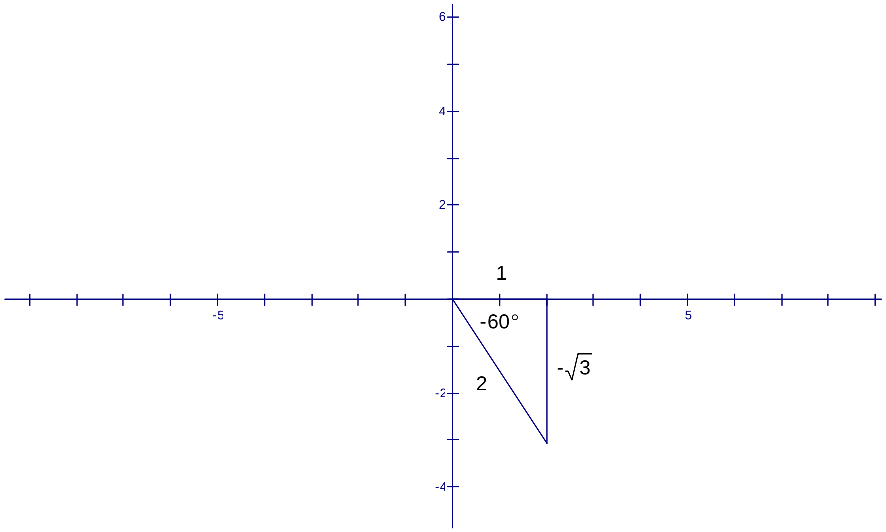
$$\Rightarrow \sin y = \frac{\sqrt{3}}{2} \; on \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \; y = 60° \; or \; y = \frac{\pi}{3}$$



2) Evaluate $\arctan(2)$

$$\arctan(-\sqrt{3})$$
$$\Rightarrow \tan y = -\sqrt{3}$$
$$\Rightarrow y = -\frac{\pi}{3}$$



3) Evaluate $\arccos(0)$

$$\arccos(0) \Rightarrow \cos y = 0 \Rightarrow y = \frac{\pi}{2}$$

Example 11

Find the inverse of $f(x) = x^3 + 2$

First, replace $f(x)$ with y

$$y = x^3 + 2$$

Next, invert x and y

$$x = y^3 + 2$$

Solve for y

$$x = y^3 + 2$$
$$\Rightarrow x - 2 = y^3 + 2 - 2$$
$$\Rightarrow x - 2 = y^3$$
$$\Rightarrow \sqrt[3]{x-2} = \sqrt[3]{y^3}$$
$$\Rightarrow \sqrt[3]{x-2} = y$$
$$\Rightarrow f^{-1}(x) = \sqrt[3]{x-2}$$