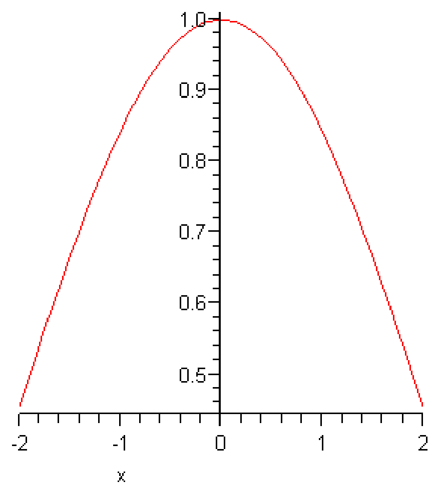
Math 151

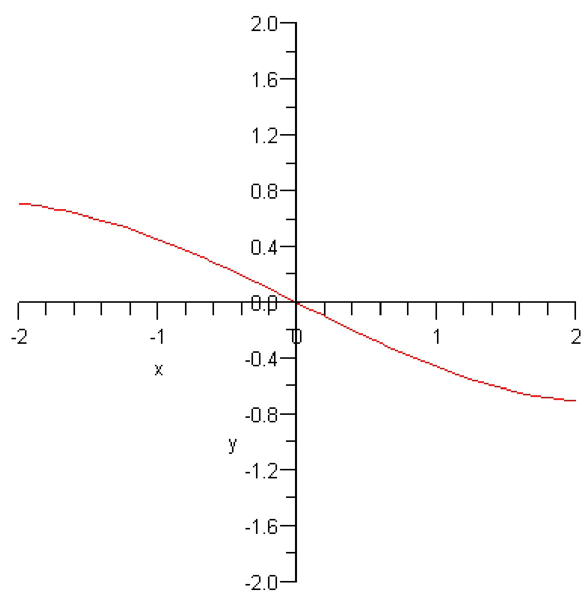
The Derivatives of Sine, Cosine, and the Exponential Function

Review: Special limits of Trigonometric Functions

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$



$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$$



The Derivative of Sine

$$f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$
$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$
$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)}{h}+\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}$$
$$= \lim_{h\to 0}\frac{\sin(x)\left(\cos(h)-1\right)}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$
$$= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$
$$= \sin(x)(0)+\cos(x)(1)$$
$$= \cos(x)$$

Rule 1: $\frac{d}{dx}\left(\sin(x)\right) = \cos(x)$

The Derivative of Cosine

$$f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$
$$= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$
$$= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\cos(x)}{h}-\lim_{h\to 0}\frac{\sin(x)\sin(h)}{h}$$
$$= \lim_{h\to 0}\frac{\cos(x)\left(\cos(h)-1\right)}{h}-\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$
$$= \cos(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$
$$= \cos(x)(0)-\sin(x)(1)$$
$$= -\sin(x)$$

Rule 2: $\frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$

The Derivative of Tangent (Will be proven in section 2.3)

$$\frac{d}{dx}\left(\tan(x)\right) = \sec^2(x)$$

The derivatives of the other trigonometric functions (Will be proven in section 2.3)

$$\frac{d}{dx}\left(\csc(x)\right) = -\csc(x) \cdot \cot(x)$$
$$\frac{d}{dx}\left(\sec(x)\right) = \sec(x) \cdot \tan(x)$$
$$\frac{d}{dx}\left(\cot(x)\right) = -\csc^2(x)$$

Example 1

Find the derivative of $y = 5\sin(x)$

$$y = 5\sin(x)$$
$$y' = 5\cos(x)$$

Example 2

Find the derivative of $y = 2x^2 + 5\cos(x)$

$$y = 2\csc(x) + 5\cos(x)$$
$$y' = \frac{d}{dx}(2x^2) + \frac{d}{dx}(5\cos(x))$$
$$y' = 4x - 5\sin(x)$$

Example 3

Find the derivative of $y = 2e^{x} + 4\sin(x)$

$y = 2e^{x} + 4\sin(x)$

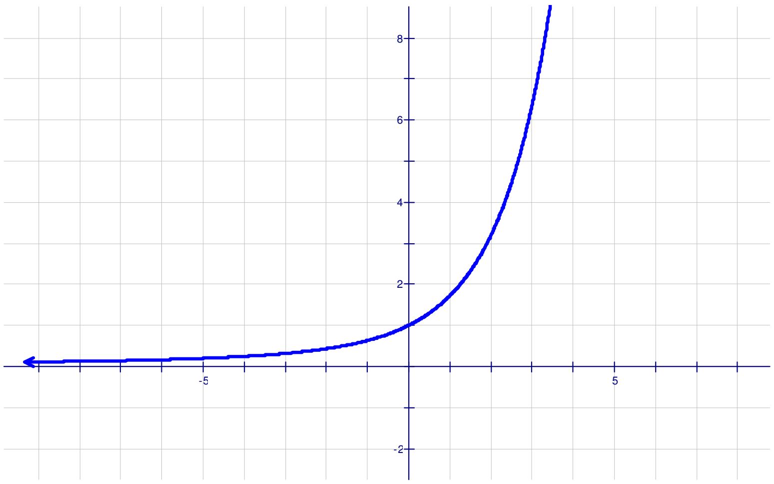
Solution:

$$y' = \frac{d}{dx}\left(2e^{x}\right) + \frac{d}{dx}\left(4\sin(x)\right)$$
$$y' = 2e^{x} + 4\cos(x)$$

The exponential function

The definition of the number e

e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$



The derivative of the exponential function

$f(x) = e^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^h}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = \lim_{h \to 0} \frac{e^x\left(e^h - 1\right)}{h}$$

$$= \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h}\right) = e^x \left[\lim_{h \to 0} \left(\frac{e^h - 1}{h}\right)\right] = e^x(1) = e^x$$

Definition: The derivative of $f(x) = e^x$ is $f'(x) = e^x$

Example 4

Find the derivative of $f(x) = 6e^x$

$f'(x) = 6e^x$

Example 5

Find the equation of a tangent line the curve $y = 2e^x$ at the point $(1,2)$

$$y = 2e^x$$
$$y' = 2e^x$$

$$m = 2e^0 = 2(1) = 2$$

$$y - y_1 = m(x - x_1)$$
$$y - 2 = 2(x - 1)$$
$$y - 2 = 2x - 2$$
$$y - 2 + 2 = 2x - 2 + 2$$
$$y = 2x$$

Example 6

Find the slope of tangent line to the function $f(x) = 2\sin(x) + 1$ at the point $(0,1)$

$$f(x) = 2\sin(x) + 1$$
$$f'(x) = 2\cos(x)$$

$$m = f'(0) = 2\cos(0) = 2(1) = 2$$

Example 7

The equation of motion of a particle is $s(t) = t^3 - 6t$ where s is in meters and t is seconds.

a) Find the velocity and acceleration as a function of t.

$$v(t) = s'(t) = 3t^{3-1} - 6t^0 = 3t^2 - 6$$

$$a(t) = s''(t) = 6t$$

b) Find the velocity after 2 seconds.

$$v(2) = 3(2)^2 - 6 = 12 - 6 = 6\ \frac{m}{s}$$

c) Find the acceleration after 2 seconds.

$$a(t) = s''(t) = 6t$$

$$a(2) = 6(2) = 12\ \frac{m}{s^2}$$