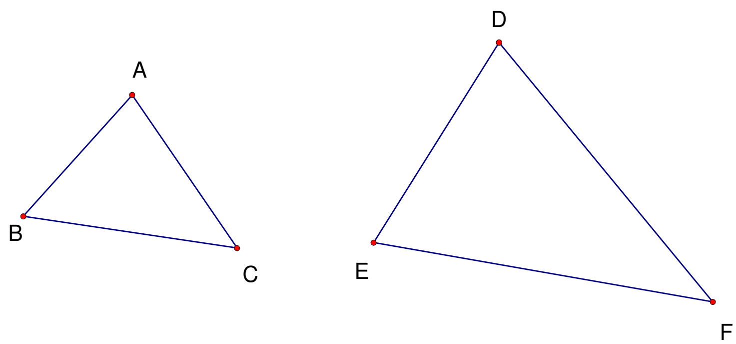
Math 135
Similar Triangles

Definition of Similar Triangles

ΔABC is **similar** to ΔDEF (written $\Delta ABC \sim \Delta DEF$) under the correspondence $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$ if and only if:

1) All three pairs of corresponding angles are congruent.
2) All pairs of corresponding sides are proportional.



$$\Delta ABC \sim \Delta DEF \;\; iff \;\; \angle A \cong \angle D : \angle B \cong \angle E : \angle C \cong \angle F \;\; and \;\; \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

AAA Similarity Postulate

Two triangles are similar if and only if three angles of one triangle are congruent, respectively, to three angles of the other triangle.

$$\Delta ABC \sim \Delta DEF \;\; iff \;\; \angle A \cong \angle D : \angle B \cong \angle E : \angle C \cong \angle F$$

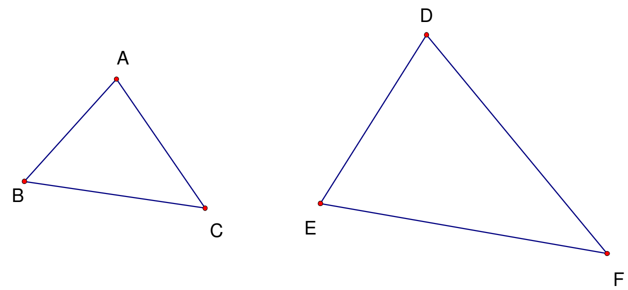
AA Similarity Theorem

Two triangles are similar if two angles of one triangle are congruent, respectively to two angles of the other triangle.

$$In \;\; \Delta ABC \;\; if \;\; \angle A \cong \angle D \;\; and \;\; \angle B \cong \angle E, \; then \;\; \Delta ABC \sim \Delta DEF$$

Proof:

Given $\angle A \cong \angle D$ *and* $\angle B \cong \angle E$, show that $\Delta ABC \sim \Delta DEF$



Let $\angle A \cong \angle D$ *and* $\angle B \cong \angle E \Rightarrow m\angle A = m\angle D$ *and* $m\angle B = m\angle E$

Since the some of the interior angles is 180 degrees, $m\angle A + m\angle B + m\angle C = 180°$ and $m\angle D + m\angle E + m\angle F = 180°$.

Use the substitution principle on the second equation with the values $m\angle A = m\angle D$ *and* $m\angle B = m\angle E$ to get the following equation. $m\angle A + m\angle B + m\angle F = 180°$.

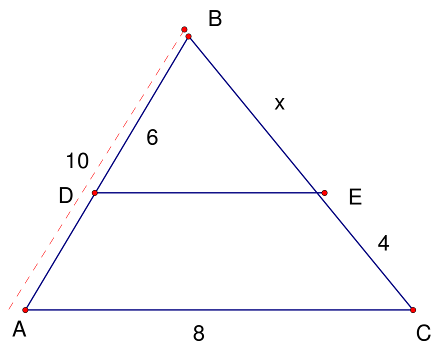
Subtracting the two equations gives:

$$\begin{aligned} &m\angle A + m\angle B + m\angle C = 180° && m\angle A + m\angle B + m\angle C = 180° \\ &-(m\angle A + m\angle B + m\angle F = 180°) \Rightarrow && \underline{-m\angle A - m\angle B - m\angle F = -180°} \\ & && m\angle C - m\angle F = 0° \\ & && m\angle C = m\angle F \\ & && \Rightarrow \angle C \cong \angle F \end{aligned}$$

Therefore $\Delta ABC \sim \Delta DEF$ by AAA Similarity Postulate

Example 1

In ΔABC, $\overline{DE}$ is parallel to $\overline{AC}$. If $AB = 10$, $CE = 4$, $AC = 8$, $BD = 6$ *and* $BE = x$, find DE and BC.



Solution:

$$\frac{6}{10} = \frac{x}{x+4}$$

Taking the product of the means and the extremes, get the following:

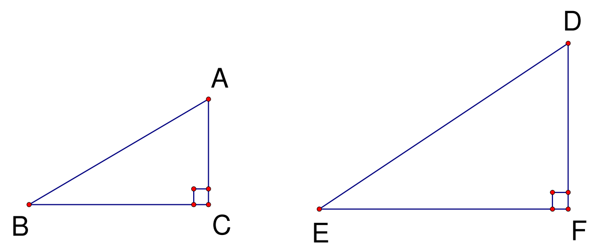
$$10x = 6(x+4)$$
$$10x = 6x + 24$$
$$10x - 6x = 6x - 6x + 24$$
$$4x = 24$$
$$\frac{4x}{4} = \frac{24}{4}$$
$$\Rightarrow x = 6 \Rightarrow BE = 6$$

Find BC

$$\frac{6}{10} = \frac{BC}{8}$$
$$10BC = 48$$
$$BC = 4.8$$

Corollary 6.5

Two right triangles are similar if an acute angle of one triangle is congruent to an acute angle of the other triangle.

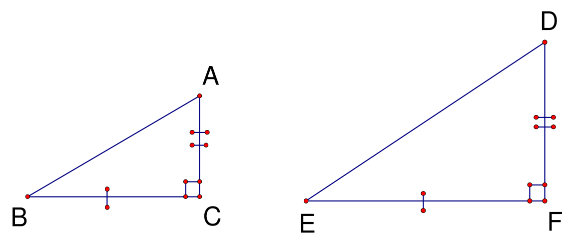


Theorem 6.6 (SAS Similarity Theorem)

Two triangles are similar if two sides are proportional, respectively, to two sides of another triangle and the angles included between the sides are congruent.

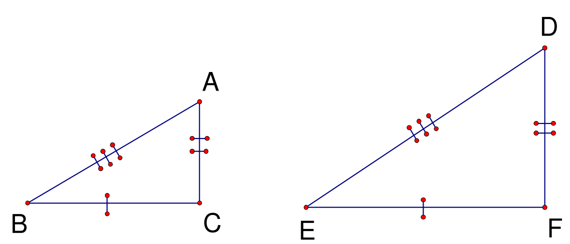
Corollary 6.7 (LL Similarity)

Two right triangles are similar if the legs of one triangle are proportional to respectively to the legs of the other triangle.



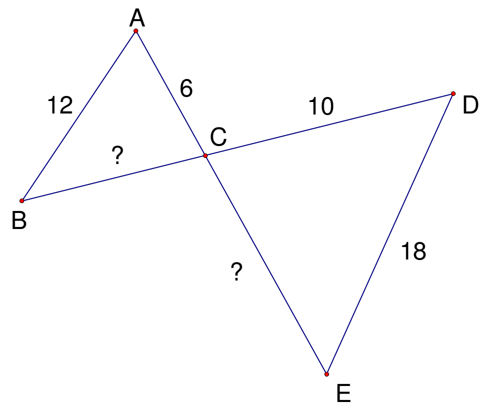
Theorem 6.8 (SSS Similarity Theorem)

Two triangles are similar if three sides of one triangle are proportional to three sides of the other triangle.



Example 2

Given $\Delta CAB \sim \Delta CED$, find the missing measures.



$$\frac{BC}{12} = \frac{10}{18}$$

$$18 \cdot BC = 12 \cdot 10$$

$$18 \cdot BC = 120$$

$$\Rightarrow BC = \frac{120}{18} = 6\frac{2}{3}$$

$$\frac{CE}{18} = \frac{6}{12}$$

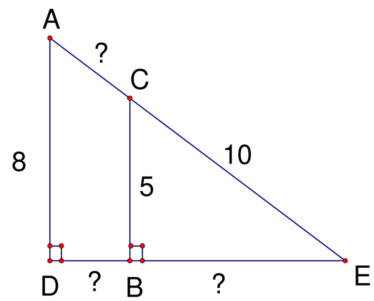
$$12 \cdot CE = 18 \cdot 6$$

$$12 \cdot CE = 108$$

$$\Rightarrow CE = \frac{108}{12} = 9$$

Example 3

Given $\Delta CAB \sim \Delta CED$, find the missing measures.



Solution:

$$\begin{aligned}
&Let\ \ AC = x \\
&\frac{10}{5} = \frac{10+x}{8} \\
&5(10+x) = 10\cdot 8 \\
&50+5x = 80 \\
&50-50+5x = 80-50 \\
&5x = 30 \\
&x = \frac{30}{5} = 6 \Rightarrow AE = 16
\end{aligned}$$

Use the Pythagorean Theorem to find BE and DB

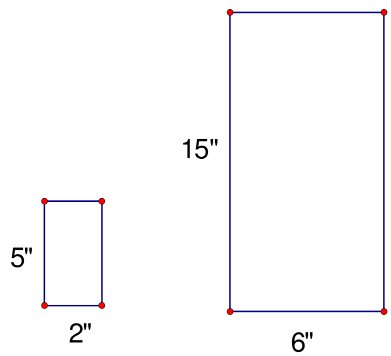
$$\begin{aligned}
&(BE)^2 + 5^2 = 10^2 && (DE)^2 + 8^2 = 16^2 \\
&(BE)^2 + 25 = 100 && (DE)^2 + 64 = 256 \\
&(BE)^2 = 75 && (DE)^2 = 192 \\
&\sqrt{(BE)^2} = \sqrt{75} && \sqrt{(DE)^2} = \sqrt{192} \\
&BE = 5\sqrt{3} && DE = \sqrt{36\cdot 3} = 6\sqrt{3}
\end{aligned}$$

Therefore, $BE = 5\sqrt{3}$ and $DB = 6\sqrt{3} - 5\sqrt{3} = \sqrt{3}$

Example 4

Compute the ratios requested in the following pairs of similar figures.

Find the ratios of base: base, height: height, and area; area for the following pair of similar rectangles.



Bases: 2:6 or 1:3

Heights: 5:15 or 1:3

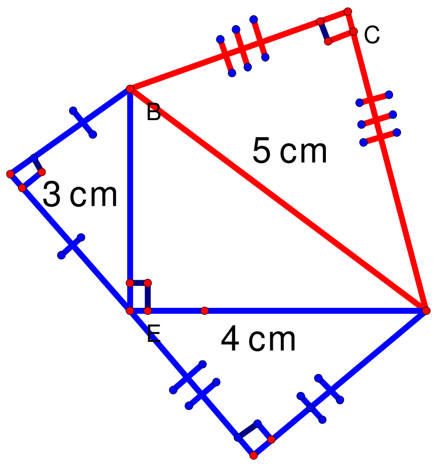
Areas: 10:90 or 1:9 (See below)

Area of smaller rectangle
$A_1 = 2 \cdot 5 = 10$

Area of larger rectangle
$A_1 = 6 \cdot 15 = 90$

Example 5

Show that the area of the larger triangle shown in red is equal to the sum of the smaller two triangles



Triangle with Hypotenuse = 3

Find the length of the leg using Pythagorean Theorem

$$a^2 + b^2 = c^2$$

$$x^2 + x^2 = 3^2$$

$$2x^2 = 9$$

$$x^2 = \frac{9}{2}$$

$$x = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}}$$

$$Length\ of\ \ leg = \frac{3}{\sqrt{2}}$$

$$Area_1 = \frac{1}{2}bh = \frac{1}{2} \cdot \frac{3}{\sqrt{2}} \cdot \frac{3}{\sqrt{2}} = \frac{9}{4}$$

Triangle with Hypotenuse = 4

Find the length of the leg using Pythagorean Theore

$$a^2 + b^2 = c^2$$
$$x^2 + x^2 = 4^2$$
$$2x^2 = 16$$
$$x^2 = \frac{16}{2}$$
$$x = \sqrt{8} = 2\sqrt{2}$$

$$Length\ of\ \ leg = 2\sqrt{2}$$
$$Area_2 = \frac{1}{2}bh = \frac{1}{2}\cdot 2\sqrt{2}\cdot 2\sqrt{2} = \frac{1}{2}\cdot 8 = 4$$

Triangle with Hypotenuse = 5

Find the length of the leg using Pythagorean Theore

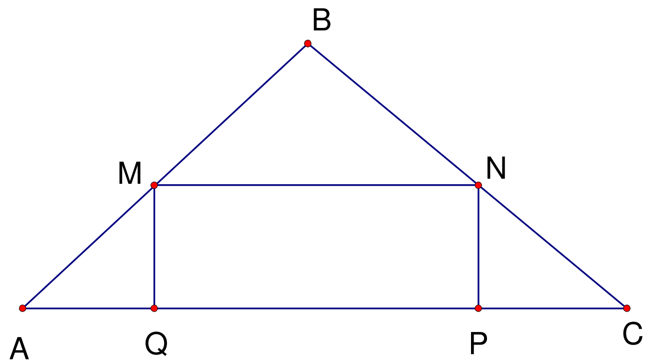
$$a^2 + b^2 = c^2$$
$$x^2 + x^2 = 5^2$$
$$2x^2 = 25$$
$$x^2 = \frac{25}{2}$$
$$x = \sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}}$$
$$Length\ of\ \ leg = \frac{5}{\sqrt{2}}$$
$$Area_3 = \frac{1}{2}bh = \frac{1}{2}\cdot\frac{5}{\sqrt{2}}\cdot\frac{5}{\sqrt{2}} = \frac{15}{4}$$

Check:

$$Area_1 + Area_2 = Area_3$$
$$\frac{9}{4} + 4 = \frac{25}{4} \Rightarrow \frac{9}{4} + \frac{16}{4} = \frac{25}{4}$$

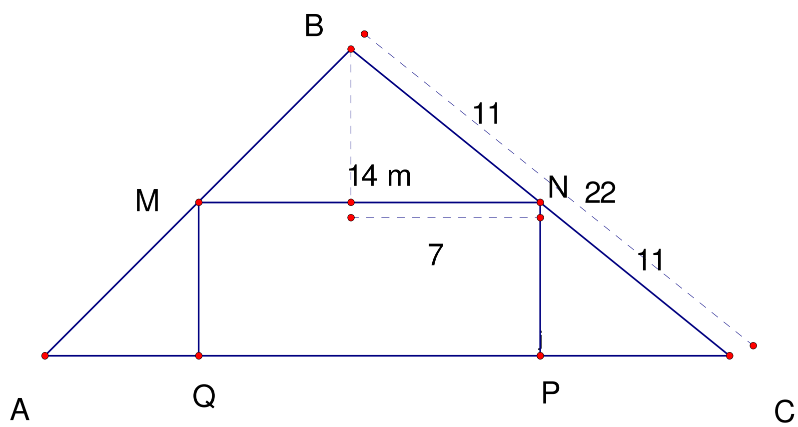
Example 6

In the figure shown, $\angle A \cong \angle C$, M and N are midpoints of the sides $\overline{AB}$ and $\overline{BC}$ and MNPQ is a rectangle . Show that $\Delta MQA \cong \Delta NPC$.



Statement	Reason
$\angle A \cong \angle C$ M and N are midpoints of the sides $\overline{AB}$ and $\overline{BC}$	Given
$CB \cong AB$	ΔABC is a isosceles triangle
$MQ \cong NP$	Opposite sides of a rectangle are congruent.
$\angle AQM$ and $\angle CPN$ are right angles	The exterior angle of a rectangle are right angles
$\angle AMQ \cong \angle CNP$	If two angles of one triangle are congruent to two angle of another triangle, then the remaining angles are congruent.
$\Delta MQA \cong \Delta NPC$	ASA

Find the area of $MNPQ$ where $MN = 14\ m$ and $BC = 22\ m$



$c^2 = a^2 + b^2$

$11^2 = 7^2 + b^2$

$121 = 49 + b^2$

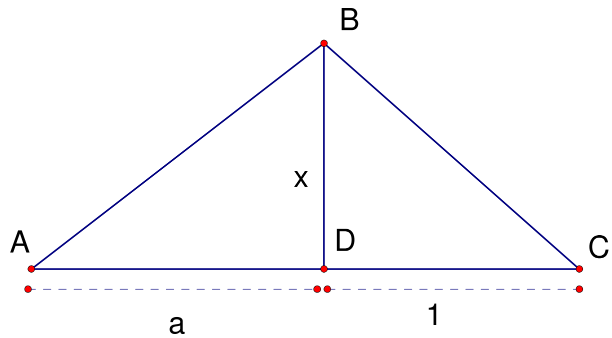
$73 = b^2 \Rightarrow b = \sqrt{72} = 6\sqrt{2}$

$NP = 6\sqrt{2}$

$Area = 14 \bullet 6\sqrt{2} = 84\sqrt{2} \approx 118.79\, m^2$

Example 7

Explain how the figure can be used to find $\sqrt{a}$



$$\frac{a}{x} = \frac{x}{1}$$

$$x \cdot x = a \cdot 1$$

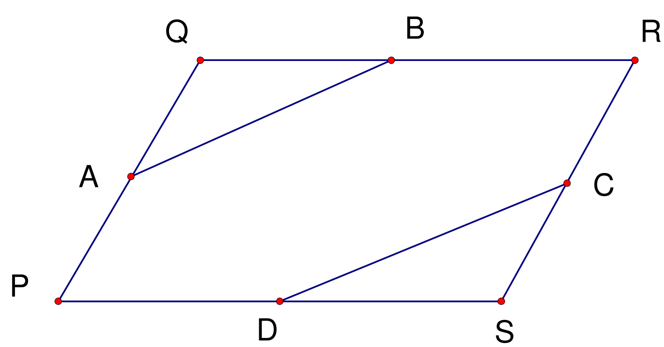
$$x^2 = a$$

$$\sqrt{x^2} = \sqrt{a}$$

$$x = \sqrt{a}$$

Example 8

In the figure shown A,B,C, and D are midpoints of the sides of parallelogram PQRS. If the area of $\Delta AQB = 5\ in^2$, find the area of hexagon PABRCD.



The parallelogram can be divided into 8 congruent triangles and the area of the hexagon can be found by adding up six of these congruent triangles. See below: