

Math 135
Networks and graphs

Key terms

Vertex (Vertices)

Each point of a graph

Edge

An edge is a segment that connects two vertices.

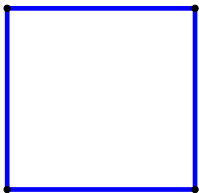
Region

A region is each individual area or separate piece of the plane that is divided up by the network.

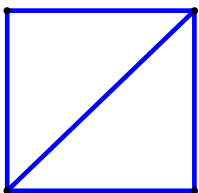
Example 1

Complete a table for the following networks.

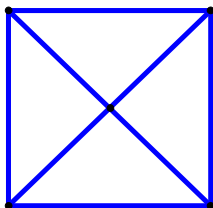
a)



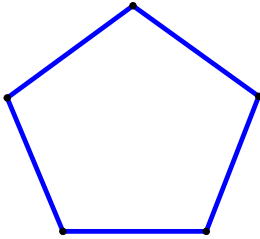
b)



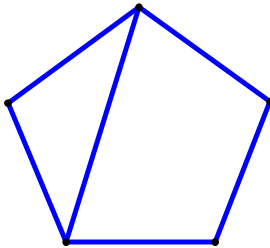
c)



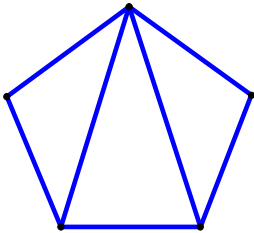
d)



e)



f)



Graph	Edges	Vertices	Regions	$V + R - 2$
A	4	4	2	$4 + 2 - 2 = 4$
B	5	4	3	$4 + 3 - 2 = 5$
C	8	5	5	$5 + 5 - 2 = 8$
D	5	5	2	$5 + 2 - 2 = 5$
E	6	5	3	$5 + 3 - 2 = 6$
F	7	5	4	$5 + 4 - 2 = 7$

A network is said to **traversable** if it can be traced in one sweep without lifting the pencil from the paper and without tracing the same edge more than once.

The **degree of a vertex** is the number of edges that meet at that vertex.

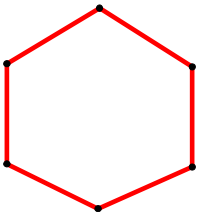
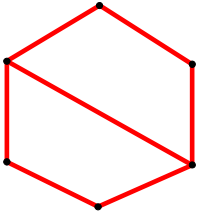
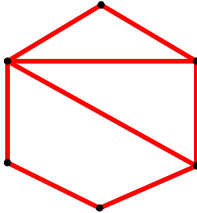
Graph	Number of edges	Degree of each vertex	Sum	Traversable
A	4	2,2,2,2	8	Yes
b	5	3,2,3,2	10	Yes
c	8	3,3,3,3,4	16	No
d	5	2,2,2,2,2	10	Yes
e	6	3,2,2,3,2	12	Yes
f	7	4,2,3,3,2	14	Yes

Rules for the number odd vertices

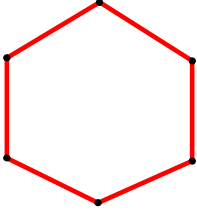
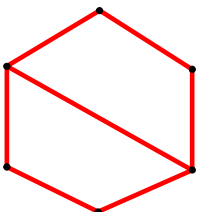
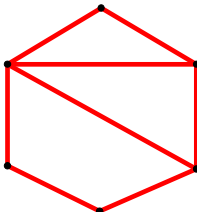
- 1) If the network has no odd vertices, then the network is traversable and any point is a starting point. The starting point will also turn out to be the ending point.
 - 2) If the network has exactly one odd vertex, then the network is not traversable. A network cannot have only one starting point or ending point without the other.
 - 3) If the network has two odd vertices, then the network is traversable. One odd vertex must be the starting point and the other odd vertex must be the ending point.
 - 4) If the network has more than two odd vertices, then the network is not traversable. A network cannot have more than one starting point and one ending point.
-

Example 1

Complete the following table

Graph	Edges	Vertices	Regions	Degree of each vertex	Sum	Traversable (Yes/No)
1) 						
2) 						
3) 						

Solution:

Graph	Edges	Vertices	Regions	Degree of each vertex	Sum	Traversable (Yes/No)
1) 	6	6	2	2,2,2,2,2,2	12	Yes
2) 	7	6	3	2,2,3,2,2,3	14	Yes
3) 	8	6	4	2,3,3,2,2,4	16	Yes

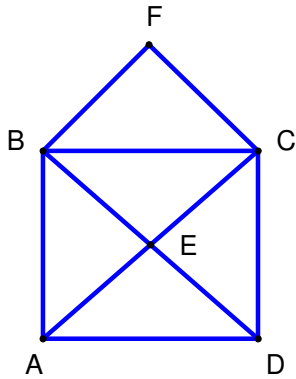
Euler Circuits

Definition

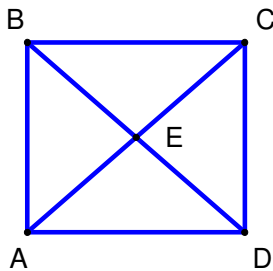
A network is an **Euler circuit** if you can start at one vertex and return to that vertex in one sweep without lifting your pencil and without tracing over the same edge more than once.

Example 2

Network 1



Network 2



Which of the following networks have an Euler circuit?

Network 1 is traversable since the graph has two odd vertices and four even vertices. (See rule 3) Vertices A and D are odd and vertices B, C, E, and F are even. However, the network does not have an Euler circuit because the path that is traversable has different starting and ending points.

Traversable path (Euler Path) is $A \rightarrow B \rightarrow F \rightarrow C \rightarrow D \rightarrow E \rightarrow B \rightarrow C \rightarrow E \rightarrow A \rightarrow D$

There is no Euler Path

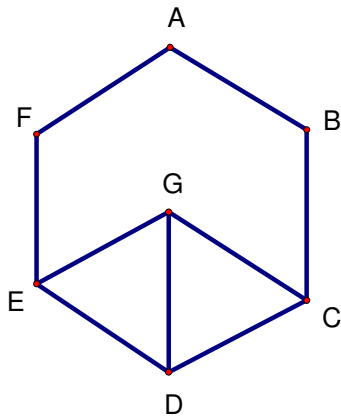
Network 2 is not even traversable because it has four odd vertices which are A, B, C, and D. Thus, the network will not have an Euler circuit.

Hamiltonian cycles

In a Hamilton cycle, you must start from a given vertex and visit each vertex only once, then return to the original vertex.

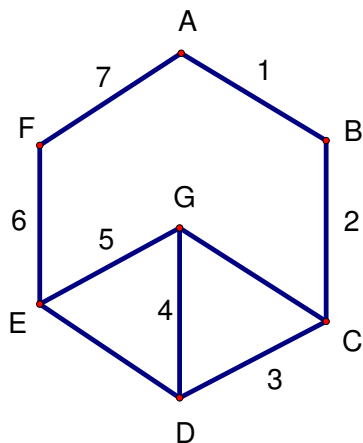
Example 3

Find the Hamiltonian cycle for the given network



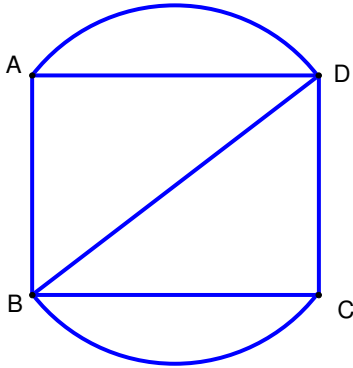
The Hamiltonian cycle would be $A \rightarrow B \rightarrow C \rightarrow D \rightarrow G \rightarrow E \rightarrow F \rightarrow A$

This pattern is shown below



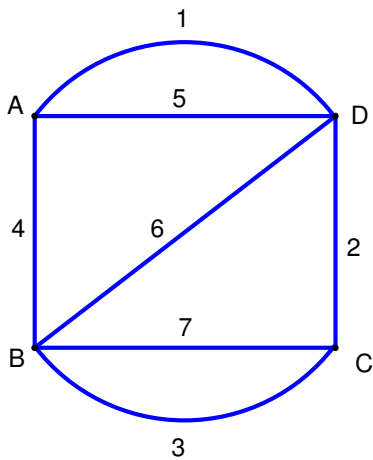
Example 4

Determine if the following graph has an Euler Circuit? Does the graph have a Hamiltonian Cycle?



The graph is traversable. (See the following Path)

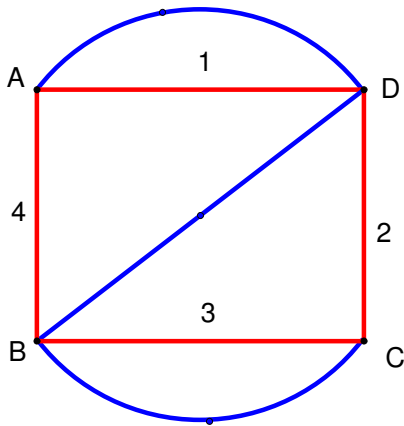
$A \rightarrow D \rightarrow C \rightarrow B \rightarrow A \rightarrow D \rightarrow B \rightarrow C$



Since the path starts and finishes at different vertices, the graph does not have an Euler path.

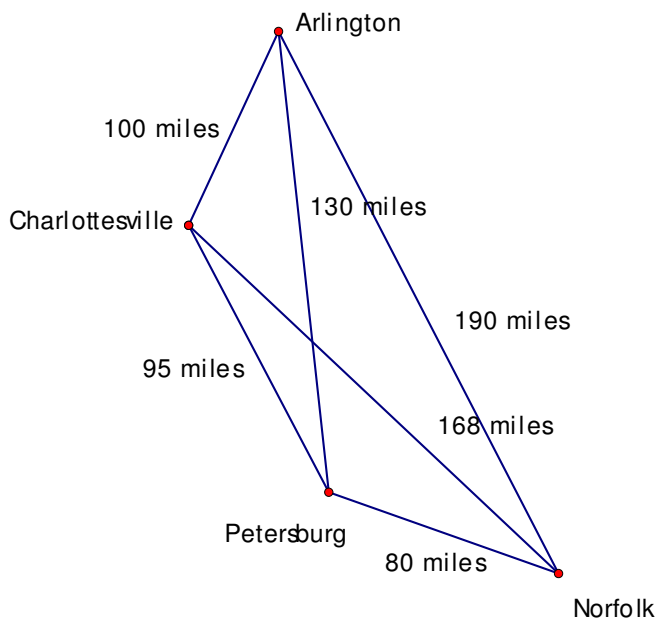
The graph does have and Hamiltonian Cycle.

$A \rightarrow D \rightarrow C \rightarrow B \rightarrow A$



The Traveling Salesman Problem

A salesman wants to visit four Virginia cities, Arlington, Charlottesville, Petersburg, and Norfolk. Driving distances are shown in figure 1. What is the shortest trip starting and ending in Arlington?



A=Arlington
N=Norfolk
P=Petersburg
C=Charlottesville

Different Paths

$$A \xrightarrow{100} C \xrightarrow{95} P \xrightarrow{80} N \xrightarrow{190} A \quad \text{Total Miles} = 465$$

$$A \xrightarrow{130} P \xrightarrow{80} N \xrightarrow{168} C \xrightarrow{100} A \quad \text{Total Miles} = 478$$

$$A \xrightarrow{190} N \xrightarrow{168} C \xrightarrow{95} P \xrightarrow{130} A \quad \text{Total Miles} = 583$$

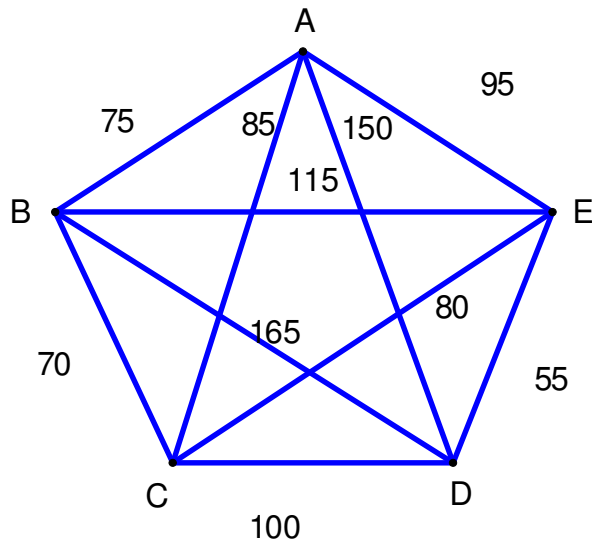
The shortest path would be

Arlington → Charlottesville → Petersburg → Norfolk → Arlington

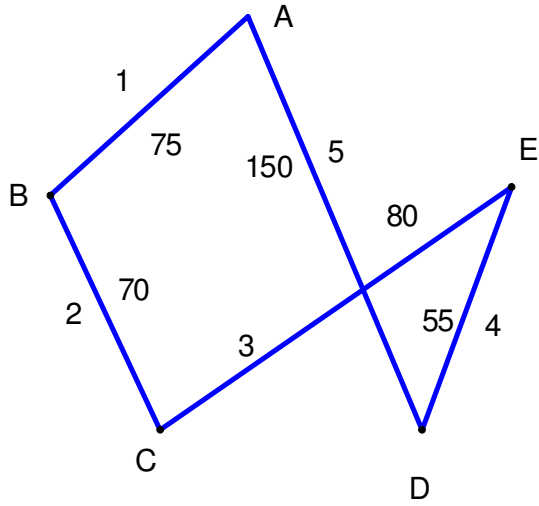
Example 4

Nearest-Neighbor Algorithm

The graph shows the mileage between various towns labeled A,B,C,D, and E. Tom, who lives in A, wishes to visit all the towns exactly once and the return home. If the sorted – edge (nearest neighbor) algorithm is used to determine an approximate solution for his best route, what is the total distance along that route?



Start by selecting the lowest weighted edge “distance” from A which is the edge AE. Now, find the lowest weighted edge from E that does not go back to A which is ED. From D the lowest weighted edge that does not go back to A is DB. At this point you have no choice but to select edges BC and AC to get back to A. (See illustration below)



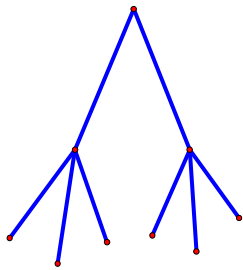
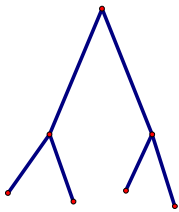
$$d = 75 + 70 + 80 + 55 + 150 = 430 \text{ miles}$$

Trees and Minimum Spanning Trees

A **tree** is a graph that is connected and has no circuits.

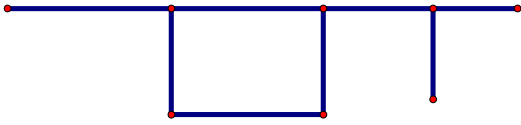
Example 5

Examples of trees

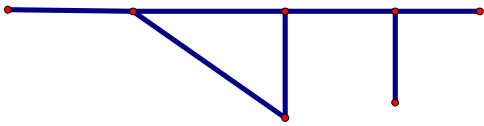


Example 6

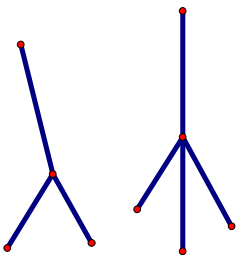
Examples of things that are not trees



This object contains a circuit.



This object contains a circuit.



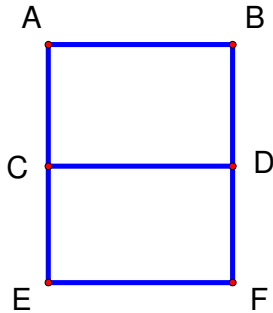
This object is not connected

Spanning Trees

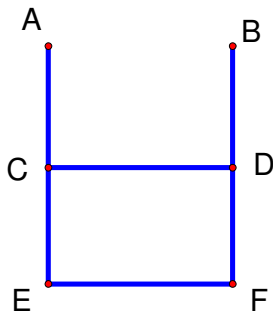
A tree that is created from another graph by removing edges but keeping the path to each vertex is called a **spanning tree**.

Example 1

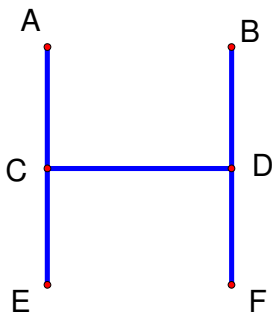
Find two spanning trees of the following graph.



First, remove all edges without eliminating any paths to each vertex. In this case we'll remove edge AB.

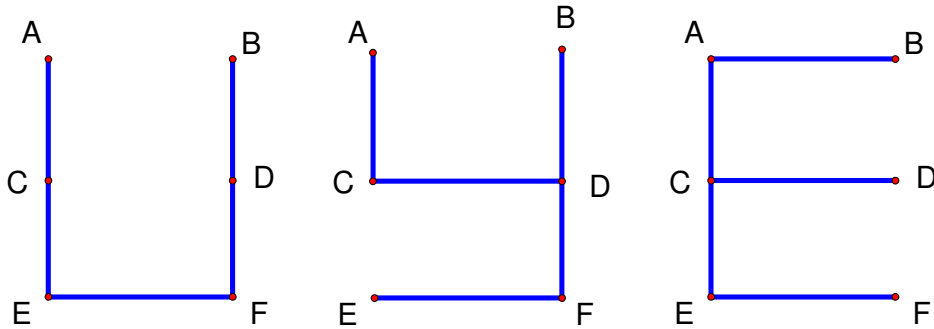


Next, remove another edge without removing a path to a vertex (Note: we cannot remove AC or BD for this reason) In this case we'll remove edge EF will form a spanning tree.



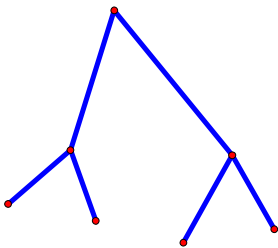
For example, removing AB will give the following graph

Here are some other spanning trees of the original graph.



If a graph is a tree with n vertices, then the number of edges is $n - 1$.

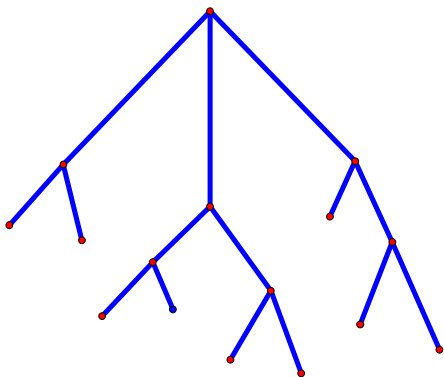
Example 7



The above graph has 7 vertices

The number of edges: $n - 1 = 7 - 1 = 6$ edges

Example 8

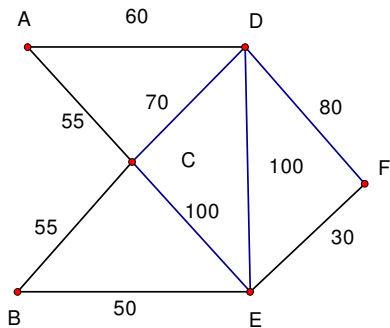


The above object has 16 vertices

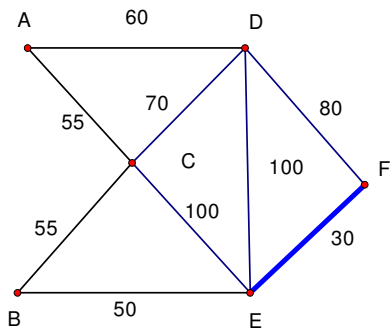
Number of edges: $n - 1 = 16 - 1 = 15$ edges

Kruskal's Algorithm

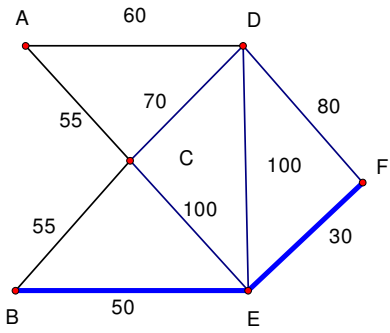
The graph below represents a number of buildings that must be connected by fiber optic cable costing \$6 per foot. The weight on the graph show the distances between the buildings. What is the cheapest way to connect the buildings.



Choose the edge of smallest weight which is EF

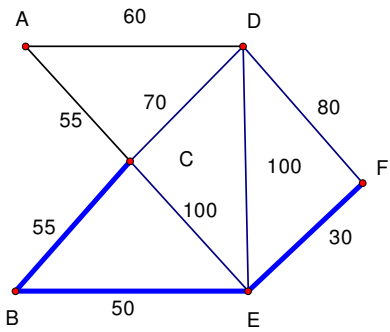


Choose the edge of smallest weight among those not yet chosen as long as it does not make a cycle.

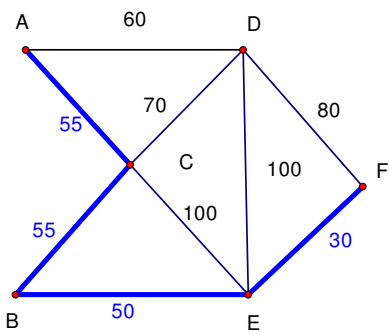


Continue this process until all vertices are connected

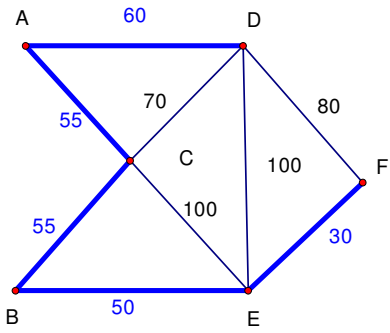
Select BC



Select AC

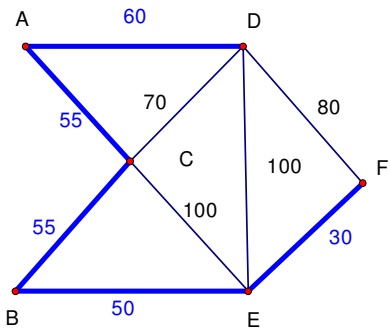


Select AD



Solution

$$30 \text{ ft} + 50 \text{ ft} + 55 \text{ ft} + 55 \text{ ft} + 60 \text{ ft} = 250 \text{ feet}$$



$$\text{Cost} = (250 \text{ ft}) \left(\frac{\$6}{\text{ft}} \right) = \$1,500.00$$