Math 135
Fractals Unit

Definition: A **fractal** is a geometric figure that is divided into smaller versions of itself.

Every fractal has an initiator and generator. The initiator represents the first step of the fractal and the generator produces each phase or step of the fractal.

Aspects of a fractal

The replacement ratio N of a fractal is the number objects generated from each object in the previous step.

The scaling ratio r of the fractal is ratio of the length of generated segment in step 1 compared to the length of initiator in step 0.

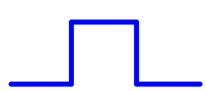
The **dimension of a fractal** is the ratio between the log of the replacement ratio to the log of s the reciprocal of the scaling ratio.

$$d = \frac{\log N}{\log s} \quad where \ s = \frac{1}{r}.$$

Example 1

The First fractal has the following initiator and generator.

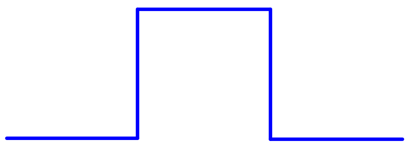
Initiator:

Generator: 

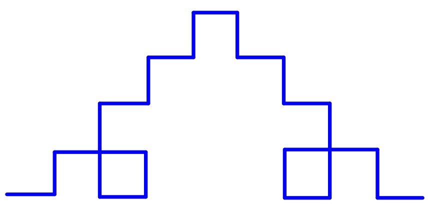
Using this generator and initiator the fractal would be generated in the following steps.

Step 0:

Step 1:



Step 2:



Aspects of the above fractal

Notice that in each step of the fractal, each segment is replaced by 5 new segments. Therefore the scaling ratio is $N = 5$

$r = \frac{1}{3}$ Each new segment form is one third the length of the segment in the previous step.

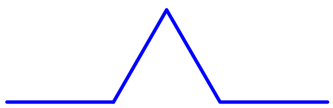
$$s = \frac{1}{r} = \frac{1}{\frac{1}{3}} = 3$$

Therefore the dimension of this fractal is $d = \frac{\log N}{\log s} = \frac{\log(5)}{\log(3)} = \frac{.699}{.477} = 1.47$

Example 2

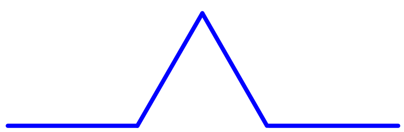
Given the initiator and generator of what is called the Koch Curve, generate the first 3 steps of the Koch Curve and find its dimension.

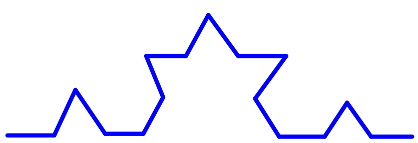
Initiator:

Generator: 

First three steps of the Koch Curve

Step 0:

Step 1: 

Step 2 

The dimension of the Koch Curve:

$N = 4$ (Each segment is replaced by 4 new segments)

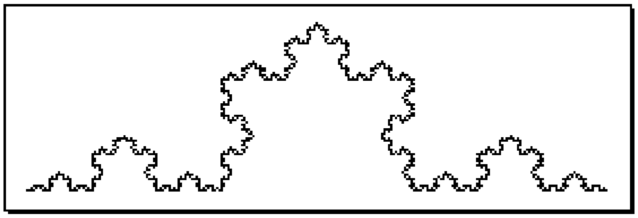
$r = \frac{1}{3}$ (Each new segment is one third the length of the previous segment)

$$s = \frac{1}{\frac{1}{3}} = 3$$

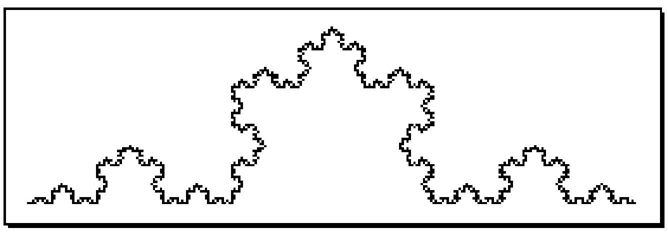
$$d = \frac{\log N}{\log s} = \frac{\log(4)}{\log(3)} = 1.26$$

Here are some graphically displayed pictures from the Internet of the 3rd and 4th step of the Koch curve:

Step 3:



Step 4:



Graphics courtesy Vanderbilt University at:

http://www.vanderbilt.edu/AnS/psychology/cogsci/chaos/workshop/Fractals.html

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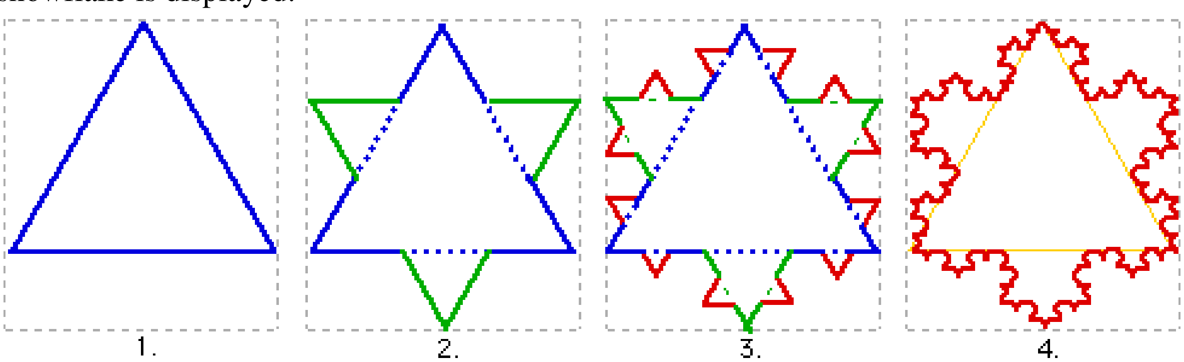
Example 3

An extension of the Koch Curve is the Koch Snowflake. If you take an equilateral triangle as the initiator and use the Koch Curve generator on each side of the equilateral triangle, the resulting fractal is known as the Koch Snowflake. In the figure below the first four steps of the Koch snowflake is displayed.



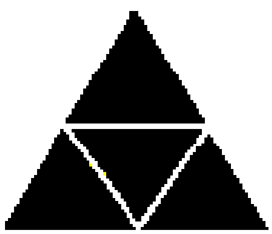
Courtesy: http://scidiv.bcc.ctc.edu/Math/snowflake.html

The dimension of the Koch Snowflake is the same as the Koch Curve.

$$N = 4 : r = \frac{1}{3} : s = 3$$
$$d = \frac{\log(4)}{\log(3)} = \frac{.602}{.477} = 1.26$$

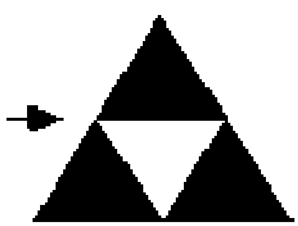
Example 4 (Sierpinski's Triangle)

Sierpinski's Triangle is created by take an equilateral triangle that is shaded solid and dividing into four equal triangles. This can be accomplished by constructing the three medians of the equilateral triangle.



After the triangle is divided into 4 smaller equilateral triangles, the middle triangle is removed as shown in the next diagram.

Step 1:



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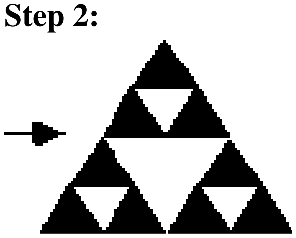
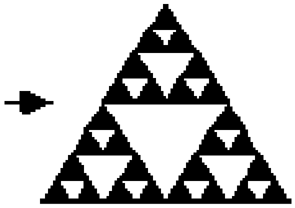
Next the above process is repeated for the remaining three shades triangles that lie in the corners of the original triangle.

Step 2:

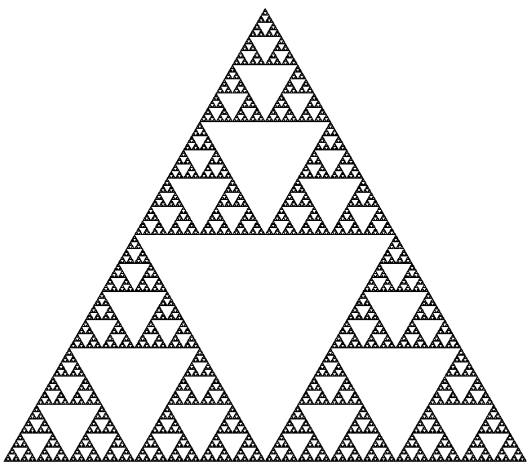


If you repeat the process in step 2, you will get the following triangle.

Step 3:



If you keep repeating this process you will generate figure shown below which is called Sierpinski's Triangle.



All diagrams are courtesy of: Cynthia Lanius: http://mathforum.org/alejandre/workshops/fractal/fractal3.html: www.google.com

The dimension of Sierpinski's Triangle

Replacement Ratio: $N = 3$ (Each triangle is replaced by three new triangles)

Scaling Ratio: $r = \frac{1}{2}$ (Each segment is divides into two segments where the new segment if one-half the original segment)

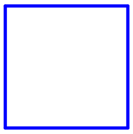
$\Rightarrow s = 2$

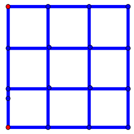
$$d = \frac{\log(3)}{\log(2)} = 1.58$$

Sierpinski's Carpet

Sierpinski's Carpet is generated by taking a square and dividing the square in 9 congruent squares, and then removing the middle square and keeping the outer 8 squares.

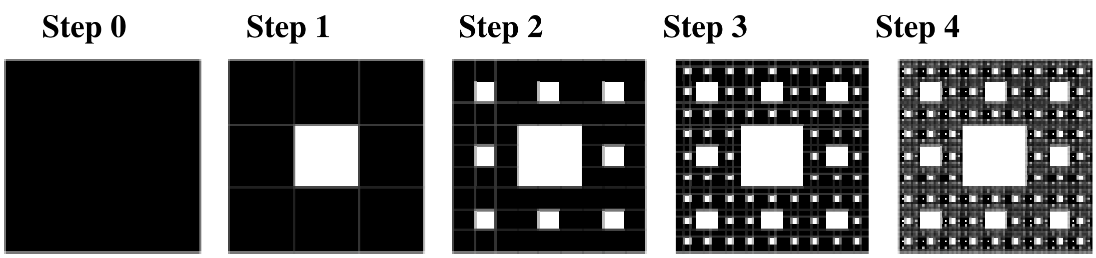
Initiator: **Generator**





In step 0 of Sierpinski's Carpet the first square generates 8 new squares, and then in step 1 each of the 8 outer squares generate 8 new squares. This process is repeated to generate the next step of the fractal. The first five steps of Sierpinski's carpet are in figure 1-1.

Figure 1-1



The dimension of Sierpinski's Carpet

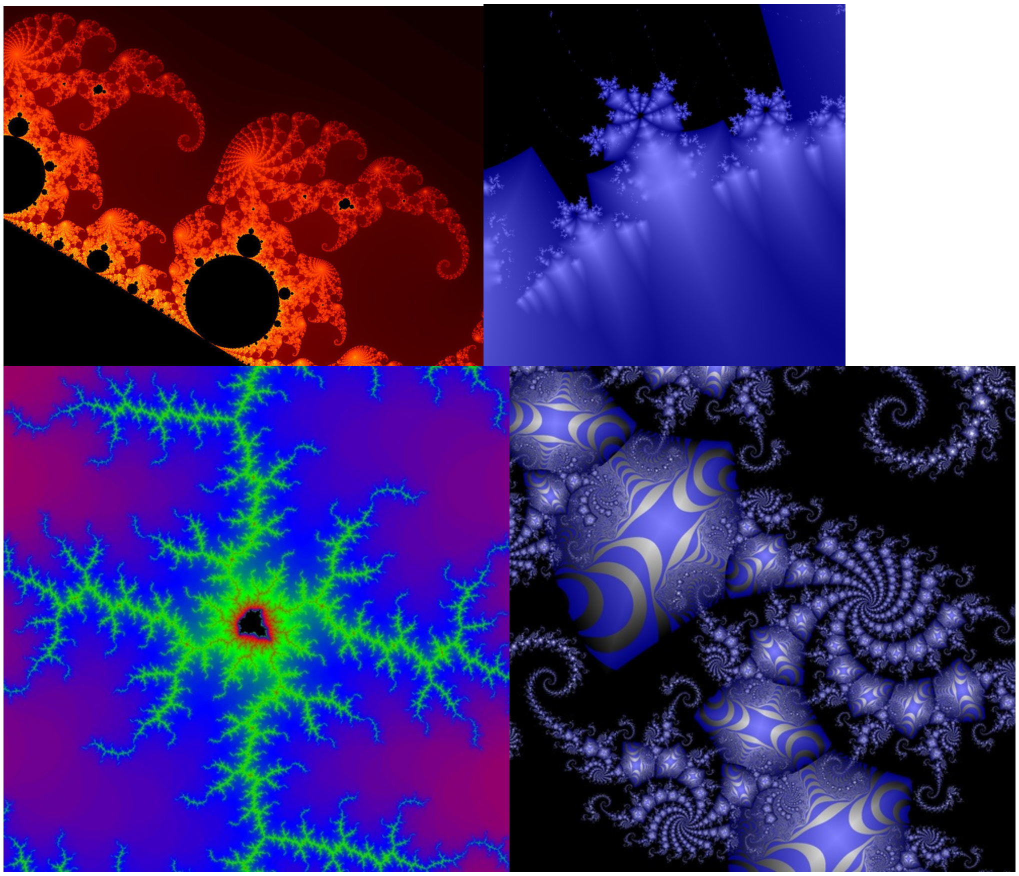
$$N = 8$$

$$r = \frac{1}{3}$$

$$s = 3$$

$$d = \frac{\log(8)}{\log(3)} = \frac{.903}{.477} = 1.89$$

Computer Generated Fractals



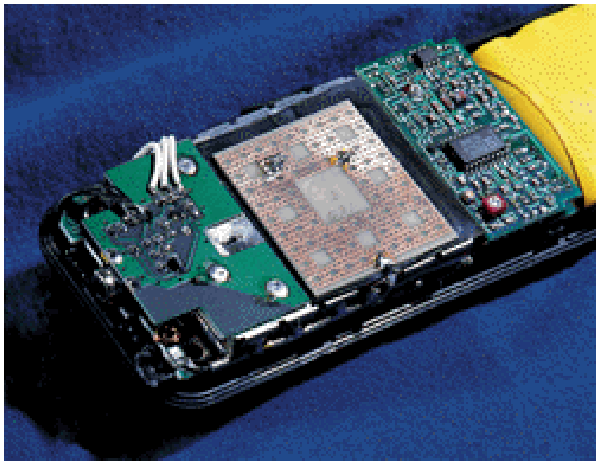
Applications of Fractals

Cellular Phone

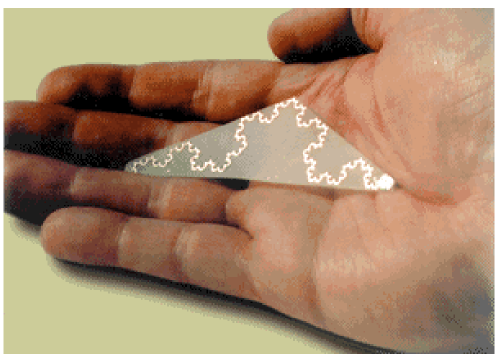
Engineer John Chenoweth discovered that fractal antennas are 25 percent more efficient than rubbery "stubby" antennas. In addition, these types of antenna are cheaper to manufacture and fractal antennas also can operate on multiple bands.

Here are some examples of fractal antennas:

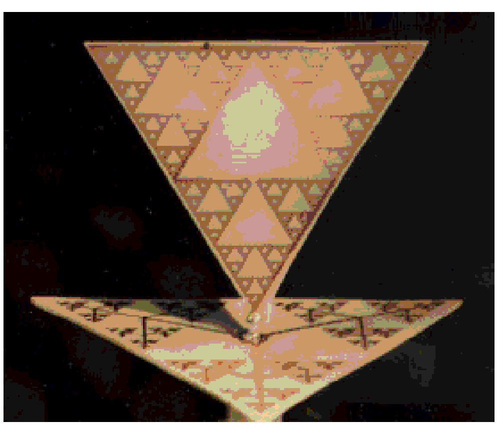
Siepinski's Carpet



Koch Curve



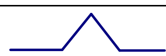
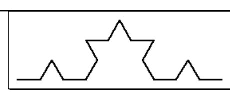
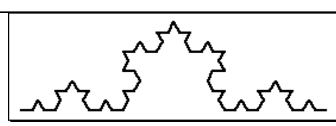
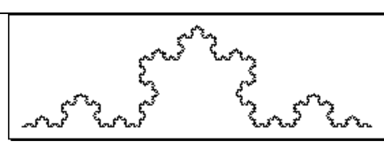
Sierpenski's Triangle



Other examples

Special Patterns

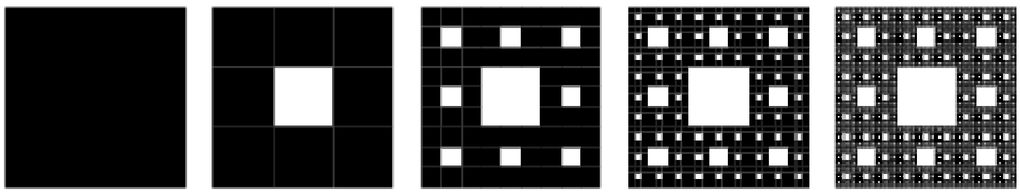
Koch Curve

Iteration	**Picture**	**Total Objects (Segments)**
0		$4^0 = 1$
1		$4^1 = 4$
2		$4^2 = 16$
3		$4^3 = 64$
4		$4^4 = 256$

$$N = 4, \quad s = 3, \quad r = \frac{1}{3}, \quad d = \frac{\log(N)}{\log(s)} = \frac{\log(4)}{\log(3)} = 1.26$$

Siepenski's Carpet

Use the space below to draw the first three iterations of Sierpenski's carpet



Iteration	**Total Objects**
0	$8^0 = 1$
1	$8^1 = 8$
2	$8^2 = 64$
3	$8^3 = 512$
4	$8^4 = 4096$

Find the dimension of Sierpenski's carpet

$$r = \frac{1}{3}, s = 3$$

$$N = 8$$

$$d = \frac{\log(8)}{\log(3)} = \frac{.9031}{.4771} = 1.89$$