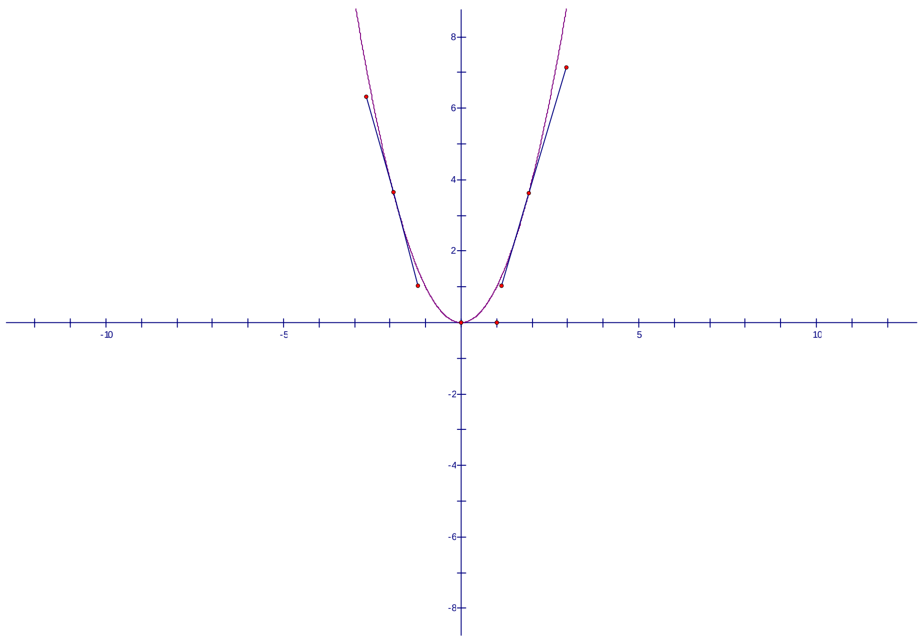
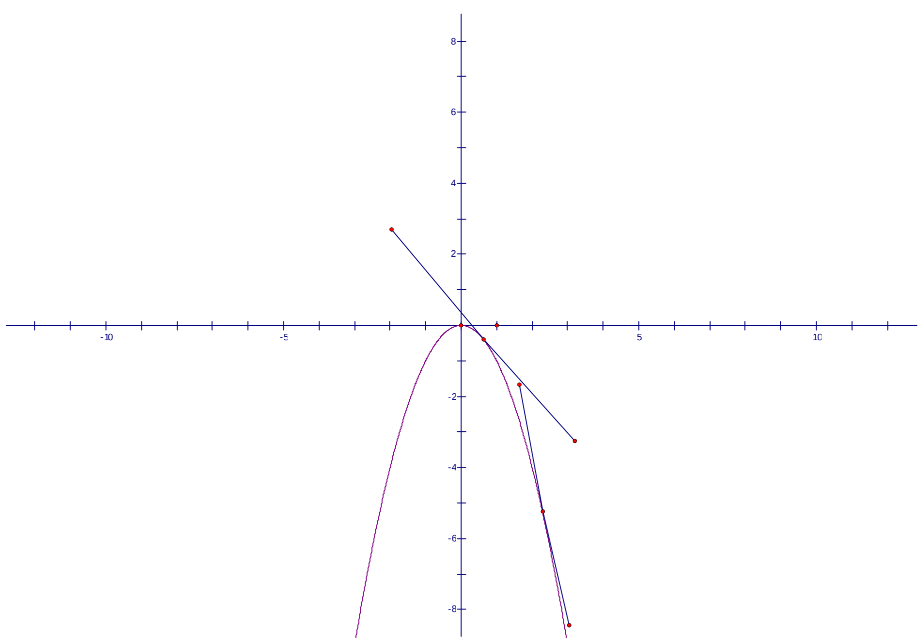
Section 3.3 Concavity

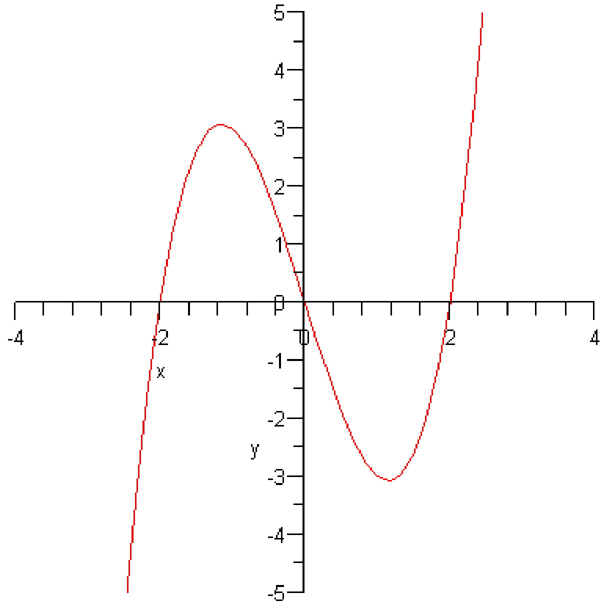
Concave Up: If a function $f(x)$ *on* I is concave up, then $f'(x)$ is increasing on I



Concave Down: If a function $f(x)$ *on* I is concave down, then $f'(x)$ is decreasing on I



Example of a graph that has both types of concavity



The graph above is concave up on $(0, \infty)$ **and concave down on** $(-\infty, 0)$

Test for Concavity

Let $f(x)$ be a function whose second derivative exist on I

1) If $f''(x) > 0$ for all x on I, then $f(x)$ is concave up.
2) If $f''(x) < 0$ for all x on I, then $f(x)$ is concave down.
3) If $f''(x) = 0$ for all x on I, then the test fails.

Example 1

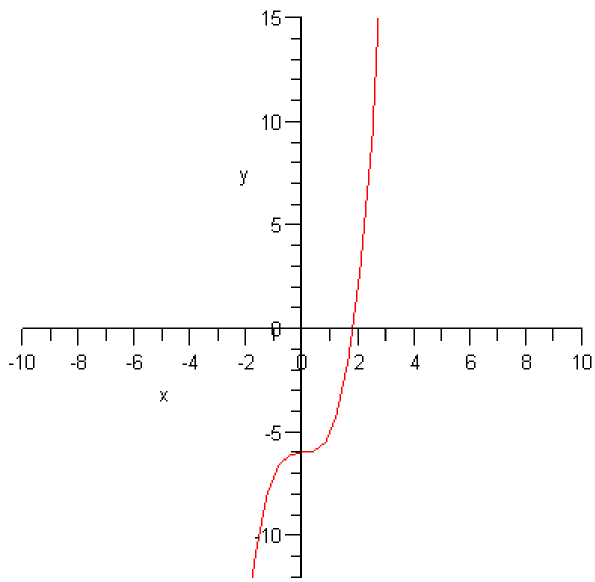
Discuss the concavity of the function $f(x) = x^3 - 6$

$$f(x) = x^3 - 6$$
$$f'(x) = 3x^2$$
$$f''(x) = 6x$$

$$6x = 0$$
$$\frac{6x}{6} = \frac{0}{6}$$
$$x = 0$$

Test for Concavity

Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	$x = -1$	$x = 1$
Sign of $f''(x)$	Negative	Positive
Conclusion	Concave Down	Concave Up



Example 2

Given the following function find all extrema points and test the concavity

$$f(x) = x^3 - 3x$$

$$f(x) = x^3 - 3x$$
$$f'(x) = 3x^2 - 3$$
$$3x^2 - 3 = 0$$
$$3(x^2 - 1) = 0$$
$$3(x-1)(x+1) = 0$$
$$(x-1)(x+1) = 0$$
$$x - 1 = 0 \ or \ x + 1 = 0$$
$$x = 1 \quad x = -1$$

$$f'(2) = 3(2)^2 - 3 = 12 - 3 = 9$$
$$f'(0) = 3(0)^2 - 3 = 0 - 3 = -3$$
$$f'(-2) = 3(-2)^2 - 3 = 12 - 3 = 9$$

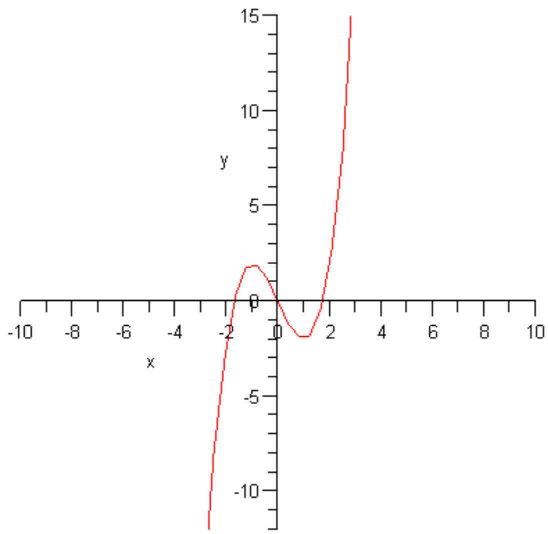
Test for Extrema points

Interval	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
Test Value	$x = -2$	$x = 0$	$x = 2$
Sign of $f'(x)$	Positive	Negative	Positive
Conclusion	Increasing	Decreasing	Increasing

y coordinates of the critical points

$f(1) = 1^3 - 3(1) = 1 - 3 = -2$

$f(-1) = (-1)^3 - 3(-1) = -1 + 3 = -2$



Thus, the function will have a relative maximum at (-12 and a relative minimum at (1,-2)

Concavity Test

$f(x) = 6x$

$6x = 0$

$\frac{6x}{6} = \frac{0}{6}$

$x = 0$

Test for Concavity

Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	$x = -1$	$x = -1$
Sign of $f''(x)$	Negative	Positive
Conclusion	Concave Down	Concave Up

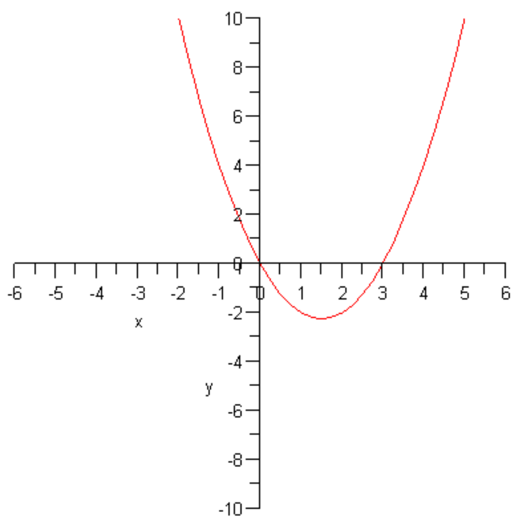
Example 3

Given the following function find all extrema points and test for concavity

$f(x) = x^2 - 2$
$f'(x) = 2x$
$2x = 0 \Rightarrow x = 0 \; Critical \; Point$

Test for Extrema Points

Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	$x = -1$	$x = 1$
Sign of $f'(x)$	Negative	Positive
Conclusion	Decreasing	Increasing



y coordinate of critical point $f(0) = 0^2 - 2 = -2$
Therefore, f has an absolute minimum at (0,2)

Concavity Test

$f''(x) = 2 > 0 \Rightarrow f \; is \; Concave \; Up \; on \; (-\infty, \infty)$

Example 4

Find all critical points and describe the concavity

$f(x) = x^4$

$f'(x) = 4x^3$

$4x^3 = 0$

$\frac{4x^3}{4} = \frac{0}{4}$

$x^3 = 0$

$\sqrt[3]{x} = \sqrt[3]{0}$

$x = 0$

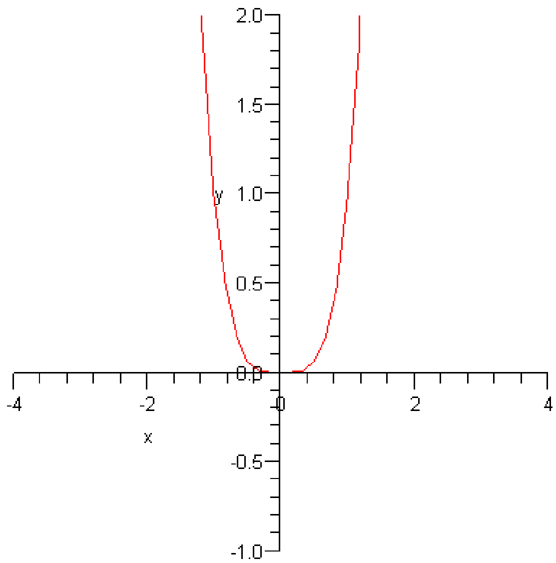
$f'(-1) = (-1)^3 = -1$

$f'(1) = 1^3 = 1$

y coordinate of critical point $f(0) = 0^4 = 0$

Test for Extrema Points

Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	$x = -1$	$x = -1$
Sign of $f'(x)$	Negative	Positive
Conclusion	Decreasing	Increasing



f has a relative min at (0,0)

Concavity Test

$$f(x) = 12x^2$$
$$12x^2 = 0$$
$$x^2 = 0$$
$$x = 0$$

$$f'(-1) = 12(-1)^2 = 12$$
$$f'(1) = 12(1)^2 = 12$$

Interval	$(-\infty,0)$	$(0,\infty)$
Test Value	$x = -1$	$x = -1$
Sign of $f''(x)$	Positive	Positive
Conclusion	Concave Up	Concave Up

Exercises Section 3.3

Given the following functions, find all extrema points and test for concavity of each function.

1) $f(x) = 5x^3 - 5$
2) $f(x) = x^3 - 3x$
3) $f(x) = x^2 - 5x$
4) $f(x) = x^3 - 3x^2$
5) $f(x) = 3x + 2$