Math 126
Section 12.2

Definite Integrals

Fundamental Theorem of Calculus

Let f be a continuous function on the closed interval $[a,b]$: then the definite integral of f exists on this interval, and

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any function such that $F'(x) = f(x)$ for all x in $[a,b]$

Example 1

Evaluate $\int_0^2 3x^2 dx$

$$\int_0^2 3x^2 dx = \frac{3}{2+1} x^{2+1} \Big|_0^2 = x^3 \Big|_0^2 = 2^3 - 0^3 = 8 - 0 = 8$$

Example 2

Evaluate $\int_1^2 4x dx$

$$\int_1^2 4x dx = \frac{4}{1+1} x^{1+1} \Big|_1^2 = \frac{4}{2} x^2 \Big|_1^2 = 2x^2 \Big|_1^2 = 2(2^2) - 1^2 = 8 - 1 = 7$$

Example 3

Evaluate $\int_0^1 (x^2 + x)dx$

$$\int_0^1 (x^2 + x)dx = \left(\frac{1}{1+2}x^{2+1} + \frac{1}{1+1}x^{1+1}\right)\Big|_0^1 = \left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_0^1 = \left(\frac{1}{3}(1)^3 + \frac{1}{2}(1)^2\right) - \left(\frac{1}{3}0^3 - \frac{1}{2}0^2\right) = \frac{1}{3} + \frac{1}{2} - 0 = \frac{5}{6}$$

Example 4

Evaluate $\int_0^2 (6x^2 + 4)dx$

$$\int_0^2 (6x^2 + 4)dx = \left(\frac{6}{1+2}x^{2+1} + \frac{4}{1+0}x^{1+0}\right)\Big|_0^2 = (3x^3 + 4x)\Big|_0^2 = (3(1)^3 + 4(2)) - (3 \cdot 0^3 - 4(0)) = (4+8) - 0 = 12$$

Example 4

Evaluate $\int_{-1}^1 (4x^3 + 2x)dx$

$$\int_{-1}^1 (4x^3 + 2x)dx$$
$$= \left(\frac{4}{1+3}x^{3+1} + \frac{2}{1+1}x^{1+1}\right)\Big|_{-1}^1$$
$$= (x^4 + x^2)\Big|_{-1}^1$$
$$= ((1)^4 + (1)^2) - ((-1)^4 + (-1)^2)$$
$$= (1+1) - (1+1)$$
$$= 2 - 2$$
$$= 0$$

Example 5

Evaluate $\int_0^1 e^x dx$

$$\int_0^1 e^x dx = e^x \Big|_0^1 = e^1 - e^0 = e - 1$$

Example 6

Evaluate $\int_2^4 \frac{1}{x} dx$

$$\int_2^4 \frac{1}{x} dx = \ln x \Big|_2^4 = \ln(4) - \ln(2)$$

Example 7

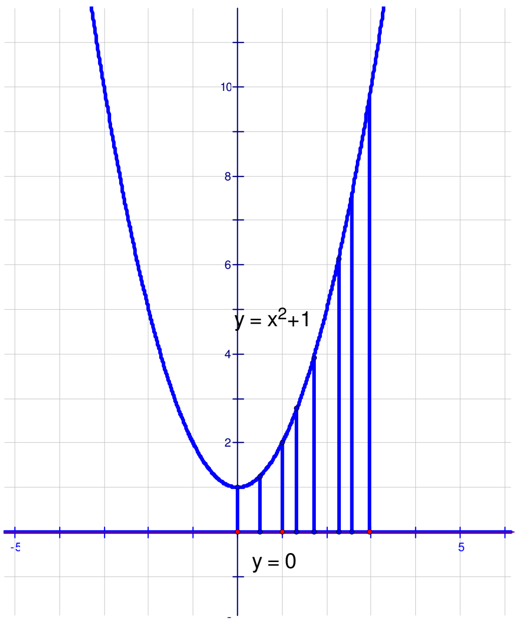
Evaluate $\int_0^1 5x^3$

$$\int_0^1 5x^3 = \frac{5}{3+1} x^{3+1} \Big|_0^1 = \frac{5}{4} x^4 \Big|_0^1 = \frac{5}{4}(1)^4 - \frac{5}{4}(0)^4 = \frac{5}{4} - 0 = \frac{5}{4}$$

Area under the curve

Example 8

Find the area between the curves $y = x^2 + 1$ and $y = 0$ on the interval $[0,3]$.



$$\int_0^3 \left(\left(x^2 + 1\right) - 0\right) dx$$

$$= \int_0^3 \left(x^2 + 1\right) dx$$

$$= \left[\frac{1}{3}x^3 + x\right]_0^3$$

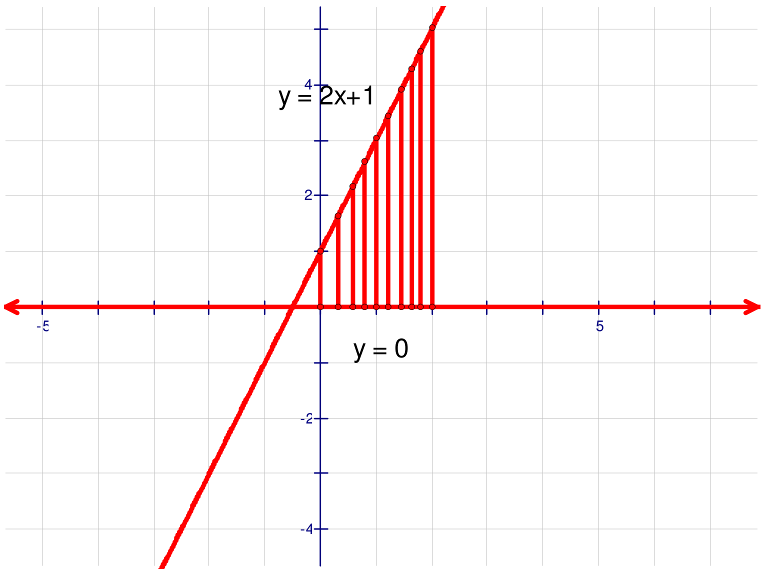
$$= \left[\frac{1}{3}(3)^3 + 3\right] - \left[\frac{1}{3}(0) + 0\right]$$

$$= 12 - 0$$

$$= 12$$

Example 9

Find the area between the curves $y = 2x + 1$ and $y = 0$ on the interval $[0,2]$.



$$\int_0^2 (2x+1-0)\,dx = \int_0^2 (2x+1)\,dx = \left[\frac{2}{1+1}x^{1+1} + \frac{1}{1+0}x^{1+0}\right]_0^2 = \left[x^2 + x\right]_0^2 = \left(2^2 + 2\right) - \left(0^2 - 0\right) = 6 - 0 = 6$$

Example 10

The marginal cost for producing *x* units of a product is modeled by $C'(x) = x^3 - x^2 + 100x$. Find the cost function, if you have a fixed cost of $200.

$$C(x) = \int (x^3 - x^2 + 100x)dx = \frac{1}{3+1}x^{3+1} + \frac{1}{2+1}x^{2+1} + \frac{100}{1+1}x^{1+1} + C = \frac{1}{4}x^4 + \frac{1}{3}x^3 + 50x^2 + C$$

The fixed cost will be a product level of zero, so use x =0 and solve for C

$$\frac{1}{4}(0)^2 + \frac{1}{3}(0)^3 + 50(0)^2 + C = 200$$
$$0 + 0 + 0 + C = 200$$
$$C = 200$$

Final solution:

$$C(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + 50x^2 + 200$$