

Appendix C

Trigonometry

Radian and Degrees

Conversion Factor $1\pi \text{ radians} = 180^{\circ}$

Converting degrees to radians

Example 1

$$120^{\circ} \cdot \frac{\pi}{180} = \frac{120\pi}{180} = \frac{2\pi}{3}$$

Example 2

Convert 45° to radians

$$45^{\circ} \cdot \frac{\pi}{180} = \frac{45\pi}{180} = \frac{\pi}{4}$$

Example 3

Convert $\frac{3\pi}{4}$ to degrees

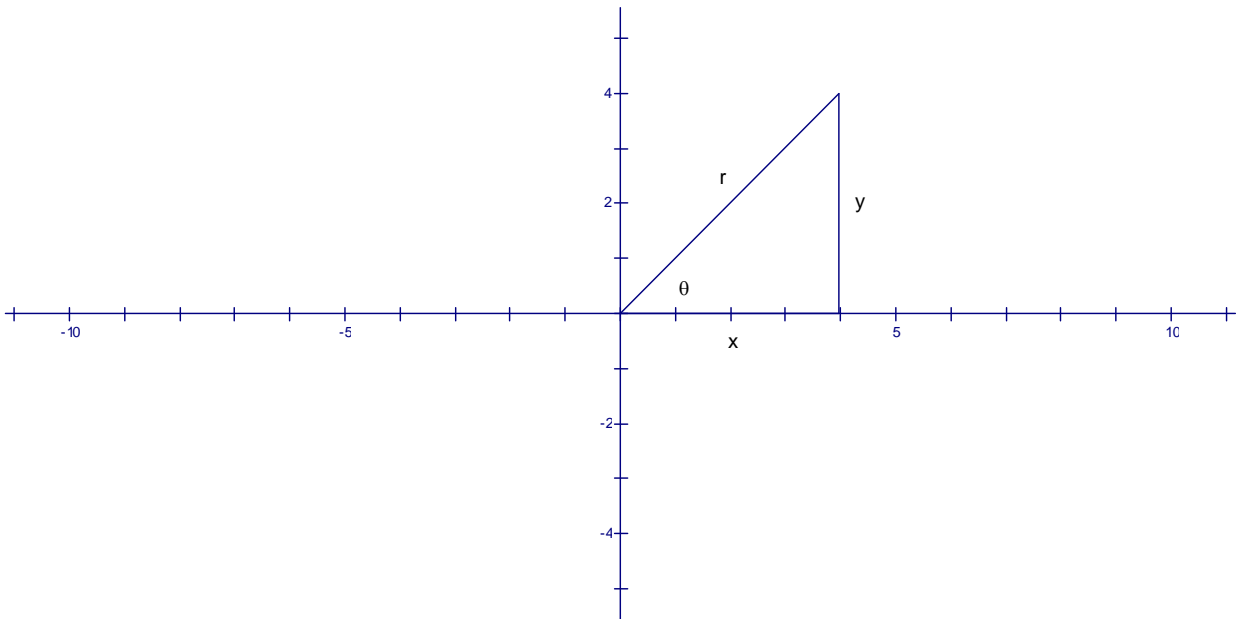
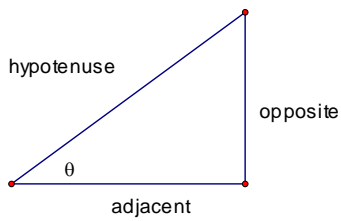
$$\frac{3\pi}{4} \cdot \frac{180}{\pi} = 3(45) = 135^{\circ}$$

Example 4

Convert $\frac{\pi}{2}$ to degrees

$$\frac{\pi}{2} \cdot \frac{180}{\pi} = \frac{180^{\circ}}{2} = 90^{\circ}$$

Basic Trigonometric Functions



$$\sin \theta = \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{y}{r}$$

$$\cos \theta = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{x}{r}$$

$$\tan \theta = \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{y}{x}$$

$$\csc \theta = \frac{\textit{hypotenuse}}{\textit{opposite}} = \frac{r}{y}$$

$$\sec \theta = \frac{\textit{hypotenuse}}{\textit{adjacent}} = \frac{r}{x}$$

$$\cot \theta = \frac{\textit{adjacent}}{\textit{opposite}} = \frac{x}{y}$$

Note: By the Pythagorean Theorem $r^2 = x^2 + y^2$

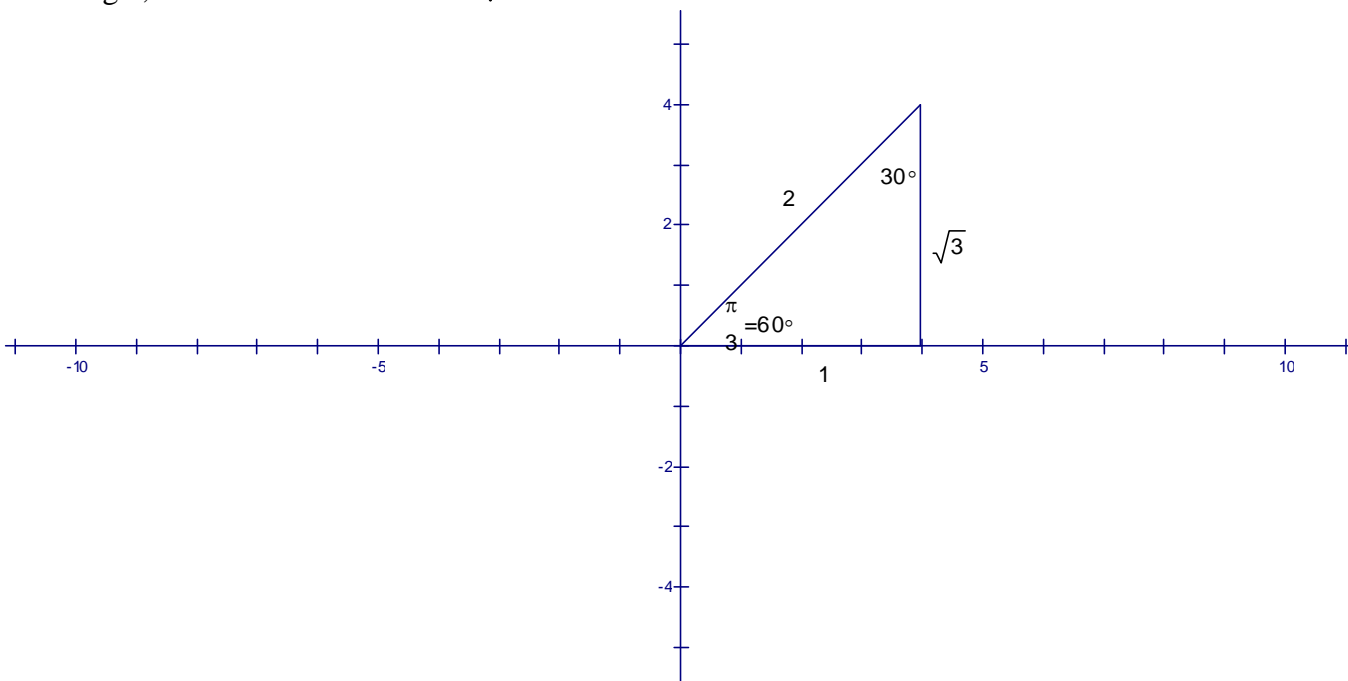
Example 5

Find exact value of the trigonometric values $\sin \theta$, $\cos \theta$, $\tan \theta$ and for the radian measure of $\frac{\pi}{3}$

First convert to degrees

$$\frac{\pi}{3} \cdot \frac{180}{\pi} = \frac{180}{3} = 60^\circ$$

Next sketch the angle in the quadrant plane. The resulting reference triangle is a 30-60-90 triangle, so the ratio of sides is $1 : \sqrt{3} : 2$



The trigonometric values are:

$$\sin \theta = \frac{y}{r} = \frac{\sqrt{3}}{2}$$

$$\cos \theta = \frac{x}{r} = \frac{1}{2}$$

$$\tan \theta = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

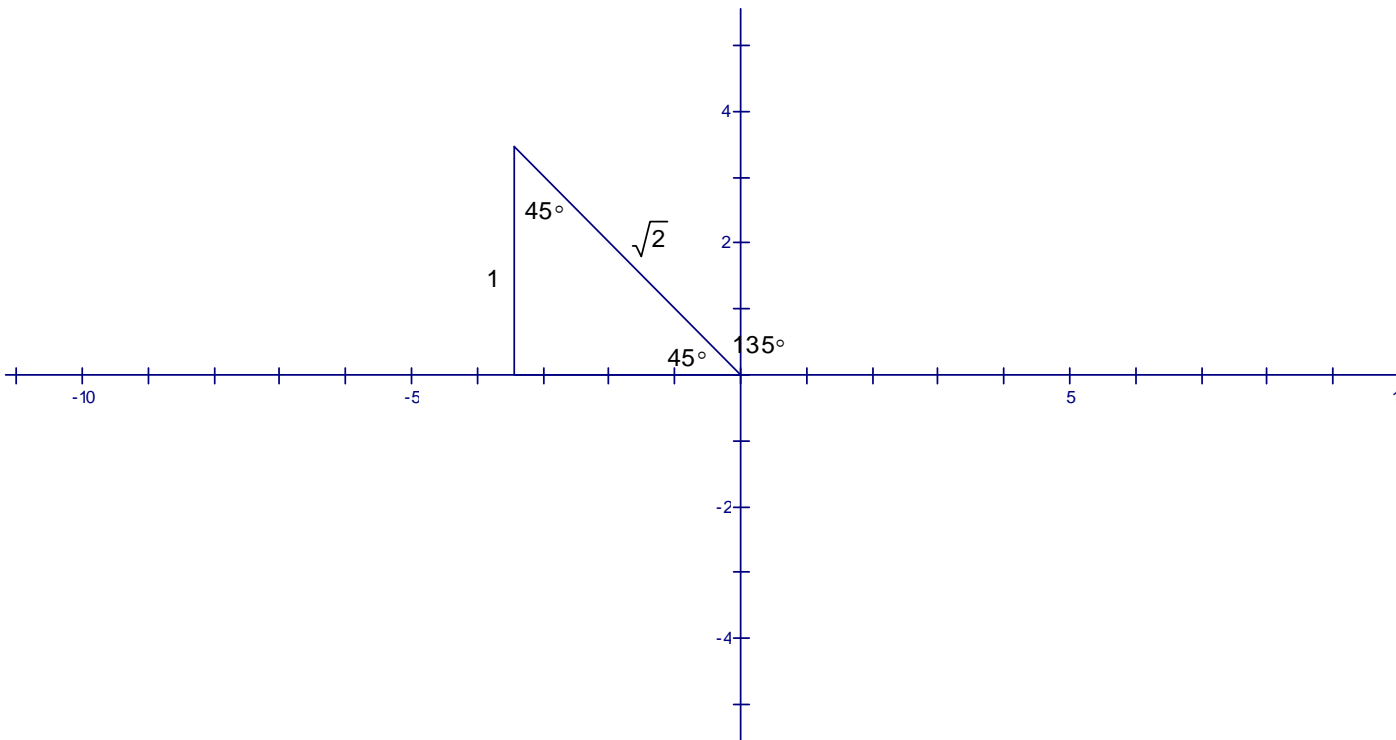
Example 6

Find exact value of the trigonometric values $\sin \theta$, $\cos \theta$, $\tan \theta$ and for the radian measure of $\frac{3\pi}{4}$

First convert to degrees

$$\frac{3\pi}{4} \cdot \frac{180}{\pi} = 3(45^\circ) = 135^\circ$$

Next sketch the angle in the quadrant plane. The resulting reference triangle is a 45-45-90 triangle, so the ratio of sides is $1:1:\sqrt{2}$



$$\sin \theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$$

$$\cos \theta = \frac{x}{r} = \frac{-1}{\sqrt{2}}$$

$$\tan \theta = \frac{y}{x} = \frac{1}{-1} = -1$$

Example 7

Find the remaining trigonometric functions given $\sin \theta = \frac{4}{5}$, in *Quadrant II*

First find the values of x , y , and r

$$\sin \theta = \frac{4}{5} = \frac{y}{r} \Rightarrow y = 4 \text{ and } r = 5$$

Use the *Pythagorean Theorem* to get x

$$r^2 = x^2 + y^2$$

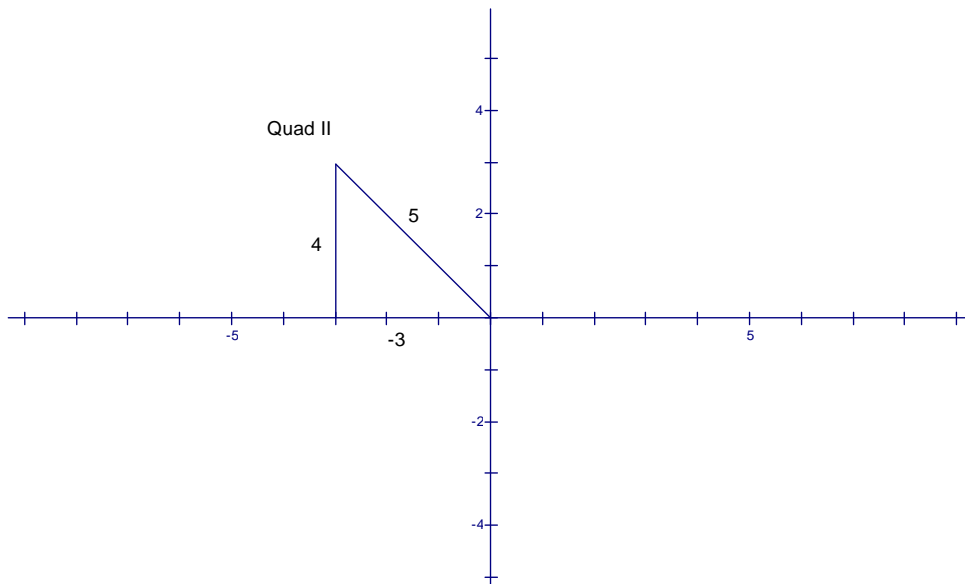
$$5^2 = x^2 + 4^2$$

$$25 = x^2 + 16$$

$$x^2 = 25 - 16$$

$$x^2 = 9$$

$$x = \pm 3$$



Since the triangle is in quadrant II x is negative, so $x = -3$

$$\cos \theta = \frac{x}{r} = -\frac{3}{5}; \tan \theta = \frac{y}{x} = -\frac{4}{3}$$

$$\csc \theta = \frac{r}{y} = \frac{5}{4}; \sec \theta = \frac{r}{x} = -\frac{5}{3}; \cot \theta = \frac{x}{y} = -\frac{3}{4}$$

Trigonometric Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

Prove $\sin^2 \theta + \cos^2 \theta = 1$

Proof:
$$\sin^2 \theta + \cos^2 \theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

Example 8 Show that $\frac{\tan x}{\sin x} = \sec x$

Proof:

$$\begin{aligned} \frac{\tan x}{\sin x} &= \frac{\tan x}{\sin x} \\ &= \tan x \cdot \frac{1}{\sin x} \\ &= \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x} \\ &= \frac{1}{\cos x} \\ &= \sec x \end{aligned}$$

Example 9

Prove $\frac{1}{\cos x} - \cos x = \sin x \tan x$

$$\begin{aligned}\frac{1}{\cos x} - \cos x &= \frac{1}{\cos x} - \cos x \\ &= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} \\ &= \frac{1 - \cos^2 x}{\cos x} \\ &= \frac{\sin^2 x}{\cos x} \\ &= \sin x \cdot \frac{\sin x}{\cos x} \\ &= \sin x \tan x\end{aligned}$$

Example 10

Evaluate $\sin^{-1}\left(\frac{1}{2}\right)$

$$\sin^{-1}\left(\frac{1}{2}\right) = y \Rightarrow \sin y = \frac{1}{2}$$

$$\Rightarrow y = \frac{\pi}{3} \text{ and } y = \frac{2\pi}{3}$$

Example 11

Evaluate $\arctan(-1)$

$$\arctan(-1) = \tan^{-1}(-1) = y$$

$$\tan y = -1$$

$$y = \frac{3\pi}{4} \text{ and } y = \frac{7\pi}{4}$$

Trigonometric equations**Example 12**

Solve $2\sin x - 1 = 0$

$$2\sin x - 1 = 0$$

$$2\sin x = 1$$

$$\sin x = \frac{1}{2} \Rightarrow x = 30^\circ$$

Example 13

Solve $2\cos^2 \theta = 1$

$$2\cos^2 \theta = 1$$

$$\cos^2 \theta = \frac{1}{2}$$

$$\sqrt{\cos^2 \theta} = \sqrt{\frac{1}{2}}$$

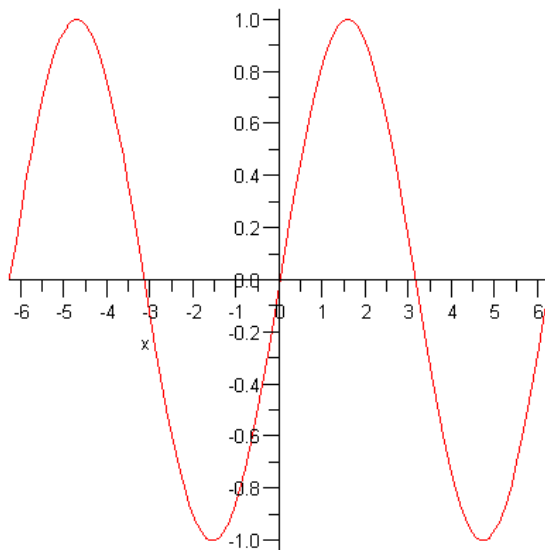
$$\cos \theta = \pm \frac{1}{\sqrt{2}} \Rightarrow \theta = 135^\circ, 315^\circ \Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Graphs of the Trigonometric Functions

Example 14

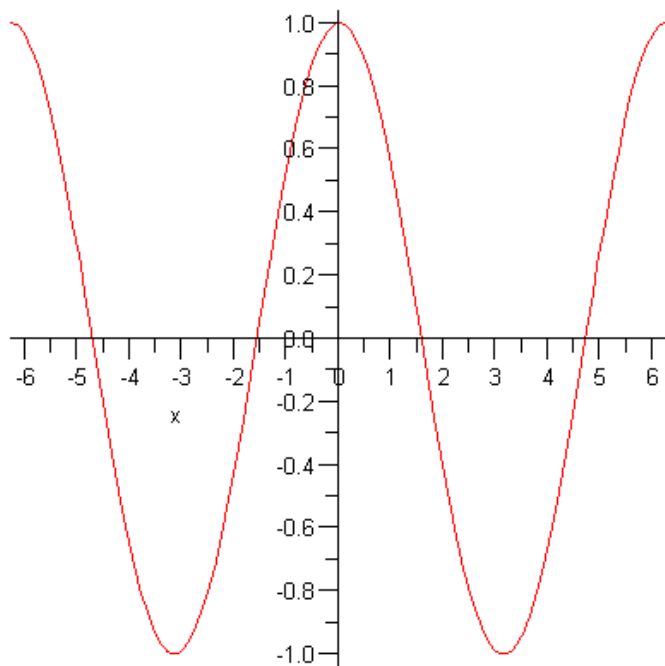
$y = \sin x$ on $[-2\pi, 2\pi]$

x	y
0	$y = \sin(0) = 0$
$\frac{\pi}{4}$	$y = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = .7$
$\frac{\pi}{2}$	$y = \sin\left(\frac{\pi}{2}\right) = 1$
$\frac{3\pi}{4}$	$y = \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} = .7$
π	$y = \sin(\pi) = 0$
$\frac{5\pi}{4}$	$y = \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} = -.7$
$\frac{3\pi}{2}$	$y = \sin\left(\frac{3\pi}{2}\right) = -1$
$\frac{7\pi}{4}$	$y = \sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2} = -.7$
2π	$y = \sin(2\pi) = 0$



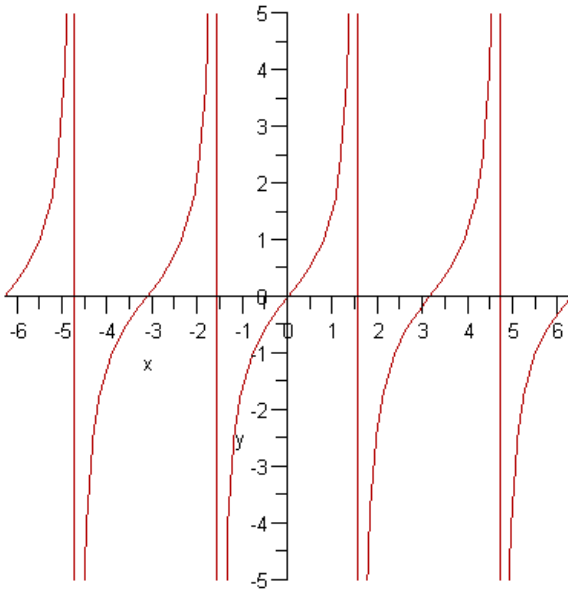
Example 15Graph $y = \cos x$ on $[-2\pi, 2\pi]$

x	y
0	$y = \cos(0) = 1$
$\frac{\pi}{4}$	$y = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = .7$
$\frac{\pi}{2}$	$y = \cos\left(\frac{\pi}{2}\right) = 0$
$\frac{3\pi}{4}$	$y = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} = -.7$
π	$y = \cos(\pi) = -1$
$\frac{5\pi}{4}$	$y = \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} = -.7$
$\frac{3\pi}{2}$	$y = \cos\left(\frac{3\pi}{2}\right) = 0$
$\frac{7\pi}{4}$	$y = \cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} = .7$
2π	$y = \cos(2\pi) = 1$



Example 15 Other graphs

$y = \tan x$ on $[-2\pi, 2\pi]$



Example 16

Graph $y = \sin\left(x - \frac{\pi}{2}\right)$ on $[-2\pi, 2\pi]$

