Exploring Quadric Surfaces with Maple

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Abstract
This paper explores some of the basic and most interesting facts about quadric surfaces. It describes the canonical coordinate transformations required to eliminate cross terms from the equation of a general quadric equation. It explains how to use these coordinates to obtain each of the seventeen canonical quadrics. It further describes the determination of the physical axis and angle of rotation. It describes the salient feature of a MAPLE 11 worksheet that can be used to analyze general quadric surfaces. This paper has several objectives. It provides instructors with a convenient technology based approach to introduce quadrics and rotations to their classes using the worksheet. At the same time, it allows us to consider several interesting mathematical topics relevant to quadric surfaces. Finally, it demonstrates that the symbolic, numerical, and graphical capabilities of a Computer Algebra System such as MAPLE 11 can be used to investigate a very complex problem in a general way to obtain important insights.
1 Introduction

There is a wealth of useful information available regarding quadric surfaces. In addition to mathematical texts that address quadric surfaces, a simple Google search leads to many useful documents that deal with various issues relevant to quadric surfaces. Unfortunately, few discuss each of the basic issues of interest to us. This paper was motivated by the desire to develop a single Maple 11 worksheet to analyze general quadric surfaces in an automatic fashion. Most of the issues discussed may be found (in one form or another) in other references. (Our favorite reference remains [9] despite its age.) However, we will have quite a bit to say about some thorny numerical issues that are not addressed in the references we have found. In addition, we will carefully sort out the issues related to the axis and angle of rotation. Finally, we will indicate several potentially interesting exercises for students.

We are interested in general quadric surfaces defined by

\[ F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gxy + 2hxy + 2px + 2qy + 2rz + d = 0. \]  (1)

As is customary, we can recast Eq. (1) as a quadratic form in the following manner. We first define the coefficient matrix

\[ A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \]

(A is equal to the Hessian of \( \frac{1}{2} F(x, y, z) \).) Eq. (1) can now be written as

\[ F(x, y, z) = <x, y, z> A(x, y, z) + <p, q, r> <x, y, z> + d. \]  (2)

The associated classification matrix is defined by

\[ A_c = \begin{pmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{pmatrix} \]

2 Basic Approach

Our first goal is to eliminate the cross terms \( xy, xz, \) and \( yz \) from Eq. (1). Properties of \( A \) and \( A_c \) determine the manner in which this may be accomplished. We will explore these properties in order to obtain the resulting canonical form for Eq. (1). We will borrow liberally from the discussion in the classical text [9]. We note that [6] contains discussions that parallel those in [9]. Note that once the canonical form for a general quadric is found it becomes a much simpler matter to obtain other quantities of interest (for example, volumes and surface areas).
Suppose the eigenvalues of $A$ are $\lambda_1, \lambda_2$ and $\lambda_3$. Let $P = (P_1, P_2, P_3)$ denote a corresponding matrix of eigenvectors. Since $A$ is symmetric, $\lambda_i$ is real and the components of $P_i$ are real. We will use $P$ to do a change of coordinates. We denote the original coordinates and the new coordinates by $X$ and $X'$, respectively, where $X = <x, y, z>^T$ and $X' = <u, v, w>^T$.

Our change of coordinates is $X = P X'$ or equivalently $X' = P^T X$. We know that $P$ diagonalizes $A$, that is, $P^T A P = D$ where $D$ is the diagonal matrix having $\lambda_i$ as the diagonal elements. The basic idea used in transforming Eq. (1) to canonical form is based on the so-called Principal Axis Theorem: if the linear terms are ignored, the second degree terms are transformed to

$$ (X')^T (P^T A P) X' = (X')^T D X' = \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2. $$

The transformed equation is thus free of the cross terms $uv$, $uw$, and $vw$. As we will in §3, the manner in which linear terms present in Eq. (1) are handled depends on the type of quadric we are dealing with.

We need to place restrictions on the eigenvector matrix to ensure it represents a proper rotation, that is, it represents an isometry that preserves distance and orientation. In order to preserve distance, the eigenvectors must be normalized to have length one. They must also be mutually orthogonal. Eigenvectors corresponding to a multiple eigenvalues are not necessarily orthogonal. We can use the Gram-Schmidt procedure to obtain a new set of mutually orthogonal eigenvectors in this case. To preserve orientation we require that the coordinate system determined by the eigenvectors is right-handed. This can be accomplished by requiring that $P_3$ is the cross product of $P_1$ and $P_2$ or equivalently, by requiring that the determinant of $P$ is equal to $+1$. For convenience in interpreting graphs of the original and canonical quadrics, we usually prefer that $P_{33} > 0$ so that the vertical axis points upward although, of course, this is not absolutely necessary. (The worksheet accomplishes this by reversing the directions of $P_3$, and possibly, $P_2$, if necessary.) A good discussion of proper rotation matrices may be found in [10].

Once the necessary modifications have been performed, we thus have a proper rotation that we can interpret as a rotation of axes. This interpretation is valid since unit vectors along the $u$, $v$, and $w$ axes are the eigenvectors of $P$ and

$$ P < 1, 0, 0 >^T = P_1 $$
$$ P < 0, 1, 0 >^T = P_2 $$
$$ P < 0, 0, 1 >^T = P_3. $$

In §4 we will discuss a second kind of rotation in which the graph of the original quadric is rotated about an axis of rotation through an angle of rotation in a certain plane to obtain the canonical quadric. Arriving at an understanding of the relationship between these two types of rotations is one of the most interesting aspects of dealing with quadric surfaces.
3 Canonical Forms

Although we are primarily interested in the real nondegenerate forms, there are seventeen different canonical forms that can arise. Table 1 is a modified version of a classic and often cited table from [9]. (See also [6].) It contains a summary of the seventeen canonical forms for quadrics and the conditions that yield them. The conditions arise naturally when the row echelon form of $A_c$ is carefully considered ([8]). In the table, the meaning of the various symbols is as follows. $\Delta = \pm 1$ or 0 depending on whether the determinant of the classification matrix $A_c$ is positive, negative, or zero. $\rho_3$ and $\rho_4$ denote the ranks of $A$ and $A_c$, respectively. $S = \pm 1$ depending on whether the eigenvalues of $P$ have the same signs (with the sign of 0 taken to be 1). Denoting the transformed constant by $d'$, we have $d' = px_0 + qy_0 + rz_0 + d$ where $X = <x_0, y_0, z_0>^T$ is any nontrivial solution of $AX = <-p, -q, -r>^T$. We define $D = \pm 1$ depending on whether the sign of $d'$ agrees with that of the eigenvalues. For types 9, 10, 15, and 16, $\Delta$ is replaced by $D$ in the table. Since we are not interested in imaginary quadrics, we will confine our attention to the real canonical quadrics summarized in Table 1.

We note that the given classification technically is ambiguous in some cases. For example, in the case of an elliptic cylinder, if $d' = 0$ the transformed quadric becomes

$$\lambda_1 u^2 + \lambda_2 v^2 = 0$$

where $\lambda_1$ and $\lambda_2$ are nonzero and have the same sign. The graph thus consists of two imaginary planes rather than an elliptic cylinder. Similarly, the equation $u^2 + v^2 = 0$ yields a line rather than a plane, and the equation $u^2 + v^2 + w^2 = 0$ yields a single point. The worksheet makes the necessary reclassification in such cases. We should point out that another precise characterization of the various quadric types in terms of the eigenvalue sign patterns for $A$ and $A_c$ is known; see [8].

Some numerical issues are noteworthy. Testing for exact zeroes isn’t reliable. The worksheet uses a floor value equal to a multiple of unit roundoff to test whether various quantities (for example, determinants, differences of near multiple eigenvalues, differences of $F(x, y, z)$ and $F(u, v, w)$, and dot products) are small enough to be considered equal to zero. We note that several of the MAPLE 11 (and presumably other Computer Algebra Systems) linear equation modules can sometimes return inconsistent results when dealing with numerically singular matrices (especially for near imaginary and other degenerate quadrics). Be sure to keep this in mind if you decide to modify the worksheet in an attempt to streamline or shorten the size of various large blocks of code that address this issue.

4 Axis and Angle of Rotation

In this section we consider a second type of rotation that represents a rotation of the graph of a quadric surface about a fixed axis of rotation. We first note that
<table>
<thead>
<tr>
<th>Quadric Type</th>
<th>ρ_3</th>
<th>ρ_4</th>
<th>Δ(orD)</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Real ellipsoid</td>
<td>3</td>
<td>4</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2 Imaginary ellipsoid</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3 Hyperboloid with one sheet</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>4 Hyperboloid with two sheets</td>
<td>3</td>
<td>4</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>5 Real cone</td>
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<td>3</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>6 Imaginary cone</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>7 Elliptic paraboloid</td>
<td>2</td>
<td>4</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>8 Hyperbolic paraboloid</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>9 Real elliptic cylinder</td>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>10 Imaginary elliptic cylinder</td>
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<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>11 Hyperbolic cylinder</td>
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<td>3</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>12 Real intersecting planes</td>
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<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>13 Imaginary intersecting planes</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>14 Parabolic cylinder</td>
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<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>15 Real parallel planes</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>16 Imaginary parallel planes</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>17 Coincident planes</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Quadric Types

a right-handed eigenvector matrix $P$ has an eigenvalue equal to 1. To see this, let $I_3$ denote the 3x3 identity matrix and consider the determinant of $P - I_3$. Observe that

$$
det(P - I_3) = det(P - PP^T) = det(P(I_3 - P)^T)$$

$$= det(P) det((I_3 - P)^T) = -det(P - I_3)$$

so that $det(P - I_3) = 0$.

Now, let $C_3$ be a nonzero vector for which $(P - I_3)C_3 = 0$. $PC_3 = C_3$ is then an eigenvector of $P$ corresponding to an eigenvalue $\mu = 1$ of $P$. The line spanned by $C_3$ is thus fixed by $P$ and represents the axis of rotation. $C_3$ need not be one of the columns of $P$ (though it sometimes is); this is a fundamental difference between 3D and 2D rotations. The rotation of interest is the rotation of the original quadric about this axis of fixed points.

We can normalize $C_3$ and extend it to an orthogonal basis $\{C_1, C_2, C_3\}$ for $\mathbb{R}^3$ for which $C_3 = C_1 \times C_2$. For convenience in interpreting plots, we usually prefer that $C_{33} > 0$ so that the axis of rotation points upward. Let $C = (C_1, C_2, C_3)$ and let $R = C^T PC$ (so that $P = CRC^T$ and $P^T = C^T RC$). The eigenvalues of $P$ are 1 and $\cos(\theta) \pm i \sin(\theta)$ and $R$ has one of two forms

$$R = \begin{pmatrix}
\cos(\theta) & \sin(\theta) & 0 \\
-\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix}$$
or

\[
R = \begin{pmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Of course, the second form is actually the same as the first except that \(\theta\) is replaced by \(-\theta\).

The rotation of the quadric about \(C_3\) occurs in the plane spanned by \(C_1\) and \(C_2\). Up to the sign of \(\theta\), the above calculation determines the angle of rotation. The worksheet needs to determine whether \(\theta\) or \(-\theta\) is appropriate. To this end we note (see [2]) that \(R\) is also equal to

\[
R_2 = I_3 + \sin(\theta) N + (1 - \cos(\theta)) N^2
\]

where

\[
N = \begin{pmatrix}
0 & c_{13} & -c_{12} \\
-c_{13} & 0 & c_{11} \\
c_{12} & -c_{11} & 0
\end{pmatrix}
\]

The worksheet uses \(R_2\) for each angle to determine which one yields \(P = CR_2CT\). In the event \(\theta < 0\) is appropriate, the worksheet replaces it by the positive angle \(2\pi + \theta\) for the sake of convenience.

As a matter of interest, we note that \(R_2\) arises very naturally due to the fact that any matrix \(R\) can be decomposed uniquely into the sum of a symmetric matrix \(S\) and a skew-symmetric matrix \(T\) for which \(R = S + T\), \(TT = -T\), \(S^T = S\), \(S = \frac{1}{2}(R + R^T)\), and \(T = \frac{1}{2}(R - R^T)\). In the present case, \(\frac{1}{2}(R + R^T) = I_3 + (1 - \cos(\theta)) N^2\) and \(\frac{1}{2}(R - R^T) = \sin(\theta) N\).

We need to say more about the issue of performing a rotation about the axis of rotation. Given the plot of the original quadric surface, if we had access to the points on the quadric surface, we could perform direct rotations (using the rotation matrix \(P^T\)). The worksheet does not do this however. The quadric surface points to be rotated are not actually embedded within the associated implicitplot3d plot structure due to the manner in which the structure is constructed and used by MAPLE 11. (Only the grid points and corresponding function values, not the actual quadric surface points, are contained in the structure.) Fortunately, however, once we know the axis of rotation \(C_3\), the angle of rotation \(\theta\), and a center or vertex \(<x_0, y_0, z_0>^T\), we can perform the necessary rotation directly using the rotate command. The following commands may be used to perform the desired rotation.

```maple
Quadric := implicitplot3d(F(x,y,z)=0, x=xmin..xmax, y=ymin..ymax, z=zmin..zmax, grid=[npts,npts,npts], axes=boxed, scaling=constrained):
plottools[rotate](Quadric, theta, [[x0,y0,z0], [x0+C3[1], y0+C3[2], z0+C3[3]]]):
```

When the above rotation is performed, MAPLE 11 constructs and uses a rotation matrix that is the same as the matrix produced by the command

```maple
rotate(Quadric, theta, [[x0,y0,z0], [x0+C3[1], y0+C3[2], z0+C3[3]]]):
```
RotM := RotationMatrix(theta, C3);

As a check on the calculations, once $\theta$ and $C_3$ are known, it is a simple matter to check that

$$\text{RotM} = CRC^T = P^T.$$ 

Curiosity about the MAPLE 11 RotationMatrix can be satisfied by viewing the source code for the command.

```maple
interface (verboseproc=3);
print (Student[LinearAlgebra][RotationMatrix]);
```

To summarize this section, the graph of the canonical quadric may be interpreted in either of two ways.

- We can rotate the original axes using the eigenvector matrix $P$.
- We can rotate the graph of the original quadric about the axis of rotation spanned by $C_3$.

Both rotations provide interesting insights about the original quadric and are useful for visualization purposes.

5 Obtaining the Canonical Forms

Quadrics may have a unique center, a line of centers, or a plane of centers. Only paraboloids and parabolic cylinders do not have at least one center. In each case, a translation to any center yields the appropriate canonical form. In the easiest case, the unique center for a quadric can be found by completing the square. (However, we opt not to complete the square exclusively to obtain a center in the general case since our approach allows us to obtain other important information in addition to the canonical equation.) Alternatively, the center is the unique solution of the linear system

$$A <x_0, y_0, z_0>^T = -<p, q, r>^T.$$

If a quadric has multiple centers, any solution of this equation can be used. Paraboloids have a single vertex and parabolic cylinders have a line of vertices. As is the case with centers, translation to any vertex yields the canonical form. If a quadric has a center, the constant term in the transformed quadric is given by

$$d' = F(x_0, y_0, z_0) = px_0 + qy_0 + rz_0 + d.$$

Principal planes are useful for obtaining the canonical forms. The equation of a principal plane relative to a system of parallel chords cutting a quadric surface in a direction $<\alpha, \beta, \gamma>$ is given by

$$(a\alpha + h\beta + g\gamma)x + (h\alpha + b\beta + f\gamma)y + (g\alpha + f\beta + c\gamma)z + (p\alpha + q\beta + r\gamma) = 0.$$
The principal planes of a quadric surface are perpendicular to and correspond to the nonzero eigenvectors of $A$, and they pass through a center or vertex. Refer to [6] and [9] for more detailed discussions of principal planes.

Following is a brief description of the manner in which the worksheet obtains the canonical form for each type of quadric.

- For cases 1, 3, 4, and 5, the unique center is located. We have
  \[ F(x, y, z) = \lambda_1 u(x, y, z)^2 + \lambda_2 v(x, y, z)^2 + \lambda_3 w(x, y, z)^2 + d' \]
  where \(<u, v, w >^T = P^T <x - x_0, y - y_0, z - z_0 >^T \).
  The canonical form used is then
  \[ H(u, v, w) = \lambda_1 (u - x_0)^2 + \lambda_2 (v - y_0)^2 + \lambda_3 (w - z_0)^2 + d'. \]

- For cases 7 and 8, the procedure is more involved. Assume the zero eigenvalue is $\lambda_3$. We use the procedure suggested in [9] (see also [6]). The equations of the two principal planes (determined by the two nonzero eigenvalues $\lambda_1$ and $\lambda_2$) are determined. The constants in the equations of these planes are $p_1$ and $p_2$ where $p_i$ is given by
  \[ p_i = \frac{P_i <p, q, r >^T}{\lambda_i}. \]
  Letting \(<u, v, w >^T = P^T <x, y, z >^T \), we obtain
  \[ F(x, y, z) = \lambda_1 (u + p_1)^2 + \lambda_2 (v + p_2)^2 + 2r'(w + p_3) \]
  where $p_3$ is yet to be determined. Denoting by $r'$ the transformed value of $r$, we find that $2r' = \pm \sqrt{-\Delta / \lambda_1 \lambda_2}$. Each of the two values for $r'$ is used to solve for $p_3$ in order to determine which value yields the quadric $F(x, y, z)$. The vertex \(<x_0, y_0, z_0 >^T \) (which is the point of intersection of the two principal planes and the tangent plane at the vertex) is then obtained by solving the system of equations
  \[
  \begin{align*}
  u(x, y, z) + p_1 &= 0 \\
  v(x, y, z) + p_2 &= 0 \\
  w(x, y, z) + p_3 &= 0
  \end{align*}
  \]
  Finally, we obtain
  \[ F(x, y, z) = \lambda_1 (u + p_1)^2 + \lambda_2 (v + p_2)^2 + 2r'(w + p_3) \]
  and
  \[ H(u, v, w) = \lambda_1 (u - x_0)^2 + \lambda_2 (v - y_0)^2 + 2r'(w - z_0). \]
• For cases 9 and 11 (elliptic and hyperbolic cylinders), we first find the line of centers determined by the solution of
\[ A <x_0, y_0, z_0 >^T = - < p, q, r >^T. \]
We note that this can be accomplished using the following command.

\[ \text{Centers} := \text{LinearSolve}(A, <-p, -q, -r>, \text{free}'t'): \]

We can then pick one of these centers using, for example, \( t = 0 \). The canonical form used is
\[ H(u, v, w) = \lambda_1 (u - x_0)^2 + \lambda_2 (v - y_0)^2 + \lambda_3 (w - z_0)^2 + d'. \]
By arranging the calculations so that \( \lambda_3 = 0 \), the canonical equation becomes
\[ H(u, v, w) = \lambda_1 (u - x_0)^2 + \lambda_2 (v - y_0)^2 + d'. \]

• For case 14 (parabolic cylinders), there is only one nonzero eigenvalue that we associate with the variable \( u \). We complete the square on \( u \) in order to obtain the canonical equation. The canonical form then consists of a sum of \( \lambda_1 u^2 \) and the remaining linear terms involving \( v \) and \( w \).

A clever alternative based on an observation in [9] may be used instead of completing the square. The principal plane determined by the nonzero eigenvalue is first determined. When \( F(x, y, z) \) is divided by an appropriate multiple of the equation of this plane, there results a linear remainder plane that is perpendicular to the principal plane and is tangent to the cylinder along the line of vertices. Put another way, the line of vertices for the original quadric is the intersection of the principal plane and the linear remainder plane. Using this fact, we can determine the line of vertices. Denoting by \( U \) the normalized equation of the principal plane and by \( W \) the normalized equation of the linear remainder plane, we obtain the canonical equation
\[ H(U, V, W) = U^2 - 2r' W = 0 \]
where \( r' \) is an appropriate constant. Although the worksheet completes the square to obtain the canonical form, it also generates this additional information.

• In the remaining cases in which the quadric consists of points, lines, or planes, the canonical quadric can be determined in a straightforward manner. Since one of these forms can result from either the absence of second degree terms in the transformed quadric or because the transformed constant leads to a degenerate form in one of the other cases, various cases are considered separately in the worksheet, specifically, the case in which there is one zero eigenvalue, the case in which there are two zero eigenvalues, and the case in which \( d' \) for one of the previous canonical forms gives rise to a degenerate equation. Interested readers are referred to the worksheet for details.
6 Examples

This section contains selected results obtained using the worksheet for several representative quadric surfaces. The worksheet contains approximately 100 other canned examples. In addition, it is possible to analyze other specific user defined quadric surfaces as well as randomly generated quadrics.

The worksheet generates various graphs, some 25 in all, of the original quadric, the canonical quadric, rotations of the original quadric, superimpositions of the original and canonical quadrics, the $P$ based eigenvector axes, the $C$ based rotation axes, rotations of the original quadric through various angles, and plots based on coordinate swapping. In each case the graphs are translated to a center or vertex of the quadric surface. All graphs are obtained using the `implicitplot3d` command with 25x25x25 grids.

A few words of explanation are in order regarding the graphs in which the original and canonical coordinates are displayed on one plot. Of course, the two quadrics are physically the same. The plot of each quadric was obtained using $xyz$ and $uvw$ coordinates, respectively. They were then simply displayed together to obtain the single plots that appear in this section. The canonical quadric is shown as a wireframe graph. In the figures the eigenvector axes are displayed along with the axis of rotation. The axis of rotation is shown in brown and the $w$ axis is shown in green. We hasten to point out that different orderings of the eigenvector axes lead to different (but equivalent) choices for the canonical coordinates and to different graphs of the canonical quadric. They also lead to a different axis and angle of rotation.

The examples in this section can be executed using the worksheet by entering the values shown in Table 2 in the first two dialog boxes when the worksheet is executed. Other canned examples as well as new ones and random ones may be run in a similar fashion as explained in the worksheet. Other dialog boxes allow experimentation with different orderings for the axes.

<table>
<thead>
<tr>
<th>Example</th>
<th>Box 1</th>
<th>Box 2</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>22</td>
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<tr>
<td>5</td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 2: Running the Examples
6.1 Example 1. Ellipsoid

Consider the ellipsoid defined by $F(x, y, z) = 0$ where

$$F(x, y, z) = 103x^2 + 125y^2 + 66z^2 - 60yz - 12xz - 48xy - 294.$$ 

The eigenvalues of $A$ are 49, 98, and 147. With this ordering of the eigenvalues (the 321 ordering in the worksheet), the canonical quadric is given by

$$H(u, v, w) = 49u^2 + 98v^2 + 147w^2 - 294.$$ 

The canonical coordinates are given by

- $u = 0.286x + 0.429y + 0.857z$
- $v = -0.857x - 0.286y + 0.429z$
- $w = 0.429x - 0.857y + 0.286z$.

The $w$-axis is spanned by the vector $<0.857, 0.429, 0.286>^T$. The axis of rotation is spanned by the vector $<0.688, -0.229, 0.688>^T$ and the angle of rotation is 249° (or −111°). Fig. 1 depicts the original and canonical ellipsoids. If the default 123 ordering is used, the axis of rotation becomes $<0, -0.894, 0.447>^T$ and the angle of rotation is 244°. Finally, if the 321 ordering is used and the $w$-axis is not forced to point upward, the axis of rotation becomes $<0.688, 0.688, 0.229>^T$ and the angle of rotation is 111° as in [9].
6.2 Example 2. Cone

Consider the cone defined by $F(x, y, z) = 0$ where

$$F(x, y, z) = 3x^2 + 3y^2 - z^2 + 2xy.$$ 

If the eigenvalues are not reordered, we obtain the eigenvalues 2, -1, and 4 along with the canonical quadric

$$H(u, v, w) = 2u^2 - v^2 + 4w^2.$$
The canonical coordinates are given by

\begin{align*}
u &= -0.707x + 0.707y \\
v &= z \\
w &= 0.707x + 0.707y.
\end{align*}

The $w$-axis is spanned by the vector $< 0.707, 1.0, 0.707 >^T$. The axis of rotation is spanned by the vector $< 0.281, 0.679, 0.679 >^T$ and the angle of rotation is $211^\circ$. Fig. 2 depicts the original and canonical cones.

![Figure 2: Original and Canonical Cones](image)

Note that if we were doing this problem by hand, we would probably use the $v$-axis as the vertical axis rather than $w$-axis. In the worksheet this may be accomplished by reordering the eigenvalues and eigenvectors. When this is done, we obtain the canonical quadric defined by

\[ H(u, v, w) = 4u^2 + 2v^2 - w^2 = 0. \]

The axis of rotation is now spanned by the vector $< 0, 0, 1 >^T$ and the angle of rotation is $315^\circ$. The canonical coordinates are given by

\begin{align*}
u &= 0.707x + 0.707y \\
v &= -0.707x + 0.707y \\
w &= z.
\end{align*}

The next figure depicts the, perhaps more pleasing, original and new canonical cones for the new ordering.
6.3 Example 3. Elliptic Paraboloid

Consider the elliptic paraboloid defined by \( F(x, y, z) = 0 \) where

\[
F(x, y, z) = x^2 + 3y^2 + z^2 + 2yz + 2xz + 2xy - 2x + 4y + 2z + 12.
\]

The vertex of this paraboloid is the point \((3, -1, -2)\). The eigenvalues are 4, 1, and 0. The canonical quadric is given by

\[
H(u, v, w) = 4(u - 3)^2 + (v + 1)^2 + 2.828(w + 2).
\]
The canonical coordinates are given by

\[ u = 0.408x + 0.816y + 0.408z \]
\[ v = -0.577x + 0.577y - 0.577z \]
\[ w = -0.707x + 0.707z. \]

The \( w \)-axis is spanned by the vector \( <0.408, -0.577, 1.0>^T \). The axis of rotation is spanned by the vector \( <-0.308, -0.594, 0.742>^T \) and the angle of rotation is 290°. Fig. 3 depicts the original and canonical paraboloids.

Figure 3: Original and Canonical Paraboloids
6.4 Example 4. Elliptic Cylinder

Consider the elliptic cylinder defined by \( F(x, y, z) = 0 \) where
\[
F(x, y, z) = 5x^2 + 2y^2 + 5z^2 - 4yz - 2xz - 4xy + 6x - 12y + 18z - 3.
\]
The eigenvalues are 6, 6, and 0. The canonical quadric is given by
\[
H(u, v, w) = 6(u - 1)^2 + 6(v - 4) - 24.
\]
The canonical coordinates are given by
\[
\begin{align*}
  u &= -0.183x - 0.365y + 0.913z \\
  v &= 0.894x - 0.447y \\
  w &= 0.408x + 0.816y + 0.408z.
\end{align*}
\]
The \( w \)-axis is spanned by the vector \( <0.913, 0.0, 0.408>^T \). The axis of rotation is spanned by the vector \( <0.516, 0.319, 0.795>^T \) and the angle of rotation is 128º. Fig. 4 depicts the original and canonical cylinders.
6.5 Example 5. Parabolic Cylinder

Consider the parabolic cylinder defined by \( F(x, y, z) = 0 \) where
\[
F(x, y, z) = 9x^2 + 36y^2 + 4z^2 - 24yz + 12xz - 36xy - 16x - 24y - 48z + 56.
\]

The eigenvalues are 49, 0, and 0. The canonical quadric is given by
\[
H(u, v, w) = 49(u - 2)^2 + 25.04(v - 1) - 50.09w + 56.
\]
The canonical coordinates are given by

\[
\begin{align*}
u &= -0.429x + 0.857y - 0.286z \\
v &= -0.894x - 0.447y \\
w &= -0.128x + 0.256y + 0.958z.
\end{align*}
\]

The \(w\)-axis is spanned by the vector \(<-0.286, -0.447, 0.958>^T\). The axis of rotation is spanned by the vector \(<-0.144, 0.089, 0.986>^T\) and the angle of rotation is 243°. Fig. 5 depicts the original and canonical cylinders.

Figure 5: Original and Canonical Cylinders

If the linear remainder plane approach discussed in §5 is used for this problem, we obtain the normalized principal plane,

\[
U = \frac{3}{7}x - \frac{6}{7}y + \frac{2}{7}z
\]

and the normalized linear remainder plane,

\[
W = \frac{2}{7}x + \frac{3}{7}y + \frac{6}{7}z - 1.
\]

(We also obtain the line of vertices \(<2 - 2t_1, 1 - \frac{2}{3}t_1, 0>\).) These yield an alternate canonical form

\[
H(U, V, W) = U^2 - \sqrt{2}W = 0
\]

whose graph matches that of \(F(x, y, z) = 0\).

7 Summary

This paper considered several of the most important questions regarding quadric surfaces. It described various techniques for finding the seventeen canonical
forms for general quadrics. It described techniques for finding the physical axis and angle of rotation. It explained the relationship between the eigenvector based rotation of axes and the rotation of the quadric about the line of fixed points of the eigenvector matrix. It described the use of a MAPLE 11 worksheet to perform the necessary symbolic and numerical calculations and to obtain various graphs of quadric surfaces. An expanded version of this paper which discusses the worksheet in more detail, contains more examples, and suggests several possible student explorations is available from the author.

References


