

Section 13.3: The Fundamental Theorem of Line Integrals

Practice HW from Stewart Textbook (not to hand in)
p. 931 # 3-9 odd, 13-21 odd

Conservative Vector Fields

A vector field \mathbf{F} is called conservative if there exists a differentiable function $f(x, y)$ where

$$\nabla f(x, y) = \mathbf{F}$$

Example 1: Given the function $f(x, y, z) = x^2 y - yz + xyz$, find the conservative vector field \mathbf{F} .

Solution:



Test for Conservative Vector Fields in the 2D Plane

The vector field $\mathbf{F}(x, y) = P \mathbf{i} + Q \mathbf{j}$ is conservative if and only if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Example 2: Determine whether or not $\mathbf{F}(x, y) = \frac{1}{y} \mathbf{i} - \frac{2x}{y^2} \mathbf{j}$ is a conservative. If it is, find a function f where $\nabla f = \mathbf{F}$.

Solution:



Example 3: Determine whether or not $\mathbf{F}(x, y) = 2xy \mathbf{i} + (x^2 - y) \mathbf{j}$ is a conservative. If it is, find a function f where $\nabla f = \mathbf{F}$.

Solution:



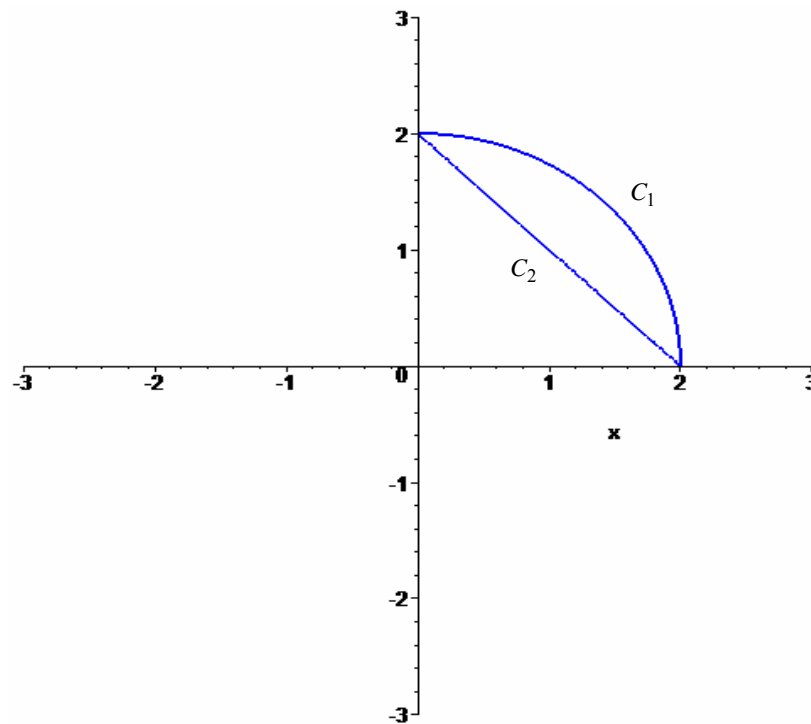
The Fundamental Theorem of Line Integrals

Consider the vector field $\mathbf{F}(x, y) = 2xy \mathbf{i} + (x^2 - y) \mathbf{j}$ and suppose we evaluate

$\int_C \mathbf{F} \cdot d\mathbf{r}$ over the two paths

$C_1 : \mathbf{r}(t) = 2\cos t \mathbf{i} + 2\sin t \mathbf{j}, 0 \leq t \leq \frac{\pi}{2}$ and $C_2 : \mathbf{r}(t) = (2 - 2t) \mathbf{i} + 2t \mathbf{j}, 0 \leq t \leq 1$.

The paths can be seen in the following diagram:



For the path $C_1 : \mathbf{r}(t) = 2\cos t \mathbf{i} + 2\sin t \mathbf{j}, 0 \leq t \leq \frac{\pi}{2}$, we saw from Example 5 in the

Section 11.2 notes that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -2$

For the path $C_2 : \mathbf{r}(t) = (2 - 2t) \mathbf{i} + 2t \mathbf{j}$, we have $x(t) = 2 - 2t, y(t) = 2t$. Thus,

$$\begin{aligned} \mathbf{F}(x(t), y(t)) &= \mathbf{F}(2 - 2t, 2t) \\ &= 2(2 - 2t)(2t) \mathbf{i} + ((2 - 2t)^2 - 2t) \mathbf{j} \\ &= (8t - 8t^2) \mathbf{i} + (4 - 4t - 4t + 4t^2 - 2t) \mathbf{j} \\ &= (8t - 8t^2) \mathbf{i} + (4 - 10t + 4t^2) \mathbf{j} \end{aligned}$$

Since $\mathbf{r}'(t) = -2 \mathbf{i} + 2 \mathbf{j}$, we have

$$\begin{aligned}
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{t=1} \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t) dt \\
&= \int_{t=0}^{t=1} \langle 8t - 8t^2, 4 - 10t + 4t^2 \rangle \cdot \langle -2, 2 \rangle dt \\
&= \int_{t=0}^{t=1} [(8t - 8t^2)(-2) + (4 - 10t + 4t^2)(2)] dt \\
&= \int_{t=0}^{t=1} (-16t + 16t^2 + 8 - 20t + 8t^2) dt \\
&= \int_{t=0}^{t=1} (24t^2 - 36t + 8) dt \\
&= (8t^3 - 18t^2 + 8t) \Big|_{t=0}^{t=1} \\
&= (8(1)^3 - 18(1)^2 + 8(1)) - 0 \\
&= -2
\end{aligned}$$

Note that for the paths C_1 and C_2 , the initial point and ending points are the same. That is each graph starts at the point $(2, 0)$ ($t = 0$ for both C_1 and C_2) and both end at the point $(0, 2)$ ($t = \frac{\pi}{2}$ for C_1 and $t = 1$ for C_2). The line integrals for both of these paths are

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -2 = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

For conservative vector fields, this common values for the line integrals is not a coincidence.

Fact: The value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if the vector field \mathbf{F} is conservative.

If we know the function f where $\nabla f(x, y) = \mathbf{F}$, there is a much easier way to evaluate to line integral of a conservative vector field \mathbf{F} , which we state next.

Fundamental Theorem of Line Integrals

Given a path $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \leq t \leq b$, with initial point $(x(a), y(a))$ and terminal point $(x(b), y(b))$. Then if \mathbf{F} is a conservative vector field with function f where $\nabla f(x, y) = \mathbf{F}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

The function has a natural extension of functions of 3 variables.

Example 4: Use the fundamental theorem of line integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for

$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$ over the path C from the point $(2, 0)$ to $(0, 2)$.

Solution:



Example 5: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}(x, y, z) = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}$ over the path

$$C : \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t^2 \mathbf{k}, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Solution:



Note: If a curve C given by $\mathbf{r}(t)$ is closed, that is, $\mathbf{r}(a) = \mathbf{r}(b)$ for $a \leq t \leq b$ where the path has the same initial and terminal points, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ if \mathbf{F} is conservative.