Section 12.2/12.3: Iterated Integrals
Double Integrals over General Regions

Practice HW from Stewart Textbook (not to hand in)
p. 842 # 1-25 odd
p. 850 # 1-21, 33-43 odd

Integration of functions with more than one variable is similar to partial differentiation. We integrate with respect to one variable and treat the other as a constant.

Example 1: Evaluate $\int_{x}^{x^2} \frac{y}{x}\, dy$.

Solution:

Example 2: Evaluate $\int_{e^y}^{y} \frac{y^2 \ln x}{x} dx$.

Solution:

$\blacksquare$

Iterated Integrals

In this section, we want to look at *iterated* integrals, which are double integrals of the form.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

or

$$\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\, dx\, dy$$

Notes

1. The inside variable of integration can be a function of the outside.
2. The outside integral must have constant limits of integration.

Example 3: Evaluate the iterated integral $\int_{1}^{2} \int_{0}^{1} (x^2 + y)\, dy\, dx$.

Solution:

∎

Example 4: Evaluate the iterated integral $\int_{0}^{2} \int_{3y^2-6y}^{2y-y^2} 4xy \, dx \, dy$.

Solution:

Note: Reversing the order of the integration variables will in most cases give the same results.

Example 5: Reverse the order of integration and evaluated the result for the iterated integral $\int_1^2 \int_0^1 (x^2 + y)\, dy\, dx$.

Solution: If you reverse the order and the limits of integration for $\int_1^2 \int_0^1 (x^2 + y)\, dy\, dx$, we obtain the integral $\int_0^1 \int_1^2 (x^2 + y)\, dx\, dy$. Then we have the following.

$$\int_0^1 \int_1^2 (x^2 + y)\, dx\, dy = \int_0^1 \left[\left(\frac{1}{3}x^3 + yx\right)\Big|_{x=1}^{x=2} \right] dy$$

$$= \int_0^1 \left[\left[\frac{1}{3}(2)^3 + y(2)\right) - \left(\frac{1}{3}(1)^3 + y(1)\right) \right] dy$$

$$= \int_0^1 \left[\frac{8}{3} + 2y - \frac{1}{3} - y \right] dy$$

$$= \int_0^1 \left(\frac{7}{3} + y\right) dy$$

$$= \left(\frac{7}{3}y + \frac{1}{2}y^2\right)\Big|_{y=0}^{y=1}$$

$$= \left(\frac{7}{3}(1) + \frac{1}{2}(1)^2\right) - \left(\frac{7}{3}(0) + \frac{1}{2}(0)^2\right)$$

$$= \frac{7}{3} + \frac{1}{2} - 0$$

$$= \frac{7}{3} \cdot \frac{2}{2} + \frac{1}{2} \cdot \frac{3}{3}$$

$$= \frac{14}{6} + \frac{3}{6}$$

$$\boxed{= \frac{17}{6}}$$

Double Integrals over Regions

For integrals of one variable, the region we integrate is always an interval. For double integrals, we want to integrate over a region *R* in the *x*-*y* plane. We denote this double integral using the notation

$$\iint_R f(x,y)\, dA$$

If $R = \{(x,y) \mid a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}$ then we write

$$\iint_R f(x,y)\, dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\, dy\, dx$$

$R = \{(x,y) \mid c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}$ then we write

$$\iint_R f(x,y)\, dA = \int_{y=c}^{y=d} \int_{x=h_1(x)}^{x=h_2(x)} f(x,y)\, dx\, dy$$

The variable of integration to apply first is usually chosen to be the one that makes the initial integration the easiest.

Example 6: Evaluate the double integral $\iint\limits_R x\cos(x^2 + 2y)\, dA$ where $R = [0, \sqrt{\pi}] \times [0, \frac{\pi}{2}]$.

Solution:

∎

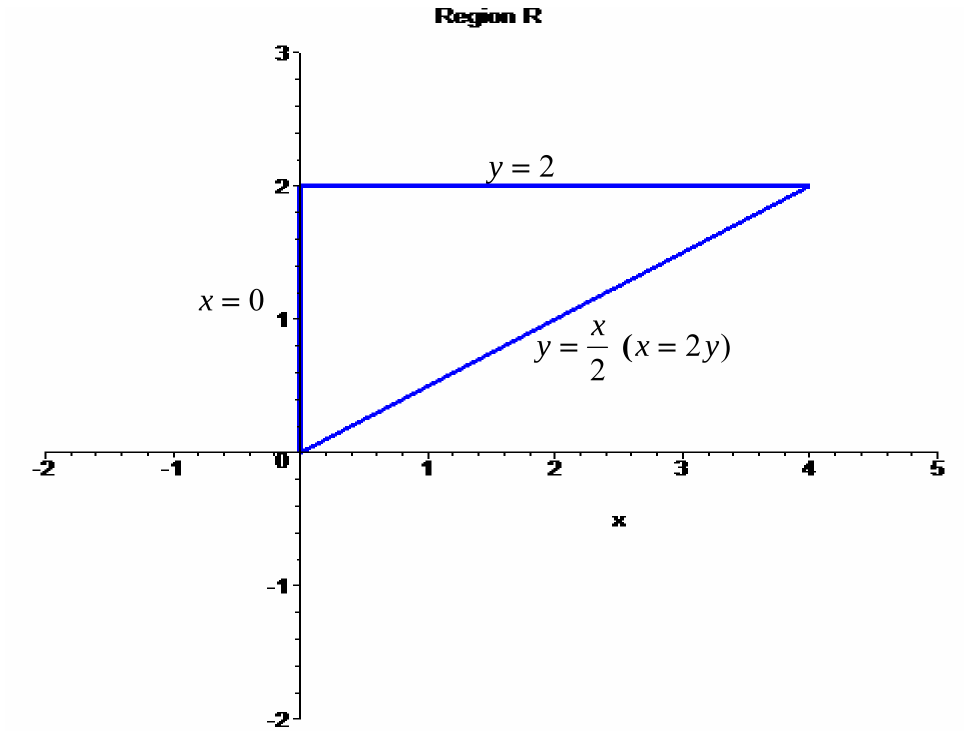
Example 7: Evaluate the double integral $\iint_R \frac{y}{1+x^2}\,dA$ where $R = \{(x, y) \mid 0 \le x \le 4 \text{ and } 0 \le y \le \sqrt{x}\}$

Solution:

$\blacksquare$

Example 8: Evaluate the double integral $\iint_R e^{-x^2}\, dA$ where $R = \left\{(x, y) \mid 0 \le x \le 4 \; and \; \frac{x}{2} \le y \le 2\right\}$

Solution: The following graph shows the region *R* outlined in blue.



If we integrate with respect to *y* first and then with respect to *x*, the double integral would be evaluated as

$$\iint_R e^{-x^2}\, dA = \int_{x=0}^{x=4} \int_{y=\frac{x}{2}}^{y=2} e^{-y^2}\, dy\, dx$$

There is no formula or method that allows one to integrate e^{-y^2} with respect to *y*. However, if we switch the order of integration and integrate with respect to *x* first, we can evaluate the integral. Since limits involving variables can only occur for the inside integral, we must use the region *R* to change the limits of integration. With respect to *x*, the region *R* changes from $x = 0$ to $x = 2y$. With respect to *y*, the region changes from $y = 0$ to $y = 2$. Thus, the double integral can be evaluated by computing the following iterated integral:

$$\iint_R e^{-x^2}\, dA = \int_{y=0}^{y=2} \int_{x=0}^{y=2y} e^{-y^2}\, dx\, dy$$

(continued on next page)

We compute this double integral as follows.

$$\iint_R e^{-x^2}\,dA = \int_{y=0}^{y=2}\int_{x=0}^{x=2y} e^{-y^2}\,dx\,dy$$

$$= \int_{y=0}^{y=2}\left[e^{-y^2}x \Big|_{x=0}^{x=2y} \right] dy \qquad \text{(With respect to } x, e^{-y^2} \text{ is treated as a constant)}$$

$$= \int_{y=0}^{y=2}\left[e^{-y^2}(2y) - e^{-y^2}(0) \right] dy \qquad \text{(Substitute in inner integration limits)}$$

$$= \int_{y=0}^{y=2} 2ye^{-y^2}\,dy \qquad \text{(Simplify)}$$

$$= -e^{-y^2}\Big|_{y=0}^{y=2}$$

Note we use $u-du$ substitution to integrate $\int 2ye^{-y^2}\,dy$

Let $u=-y^2$, $du=-2ydy$ or $-du = ydy$

Then $\int 2ye^{-y^2}\,dy = \int e^u(-du) = -e^u + C = -e^{-y^2} + C$

$$= -e^{-(2)^2} - -e^{-(0)^2} \qquad \text{(Substitute in outer integration limits)}$$

$$= -e^{-4} + 1 \qquad \text{(Simplify)}$$

$$\boxed{= 1 - e^{-4}}$$

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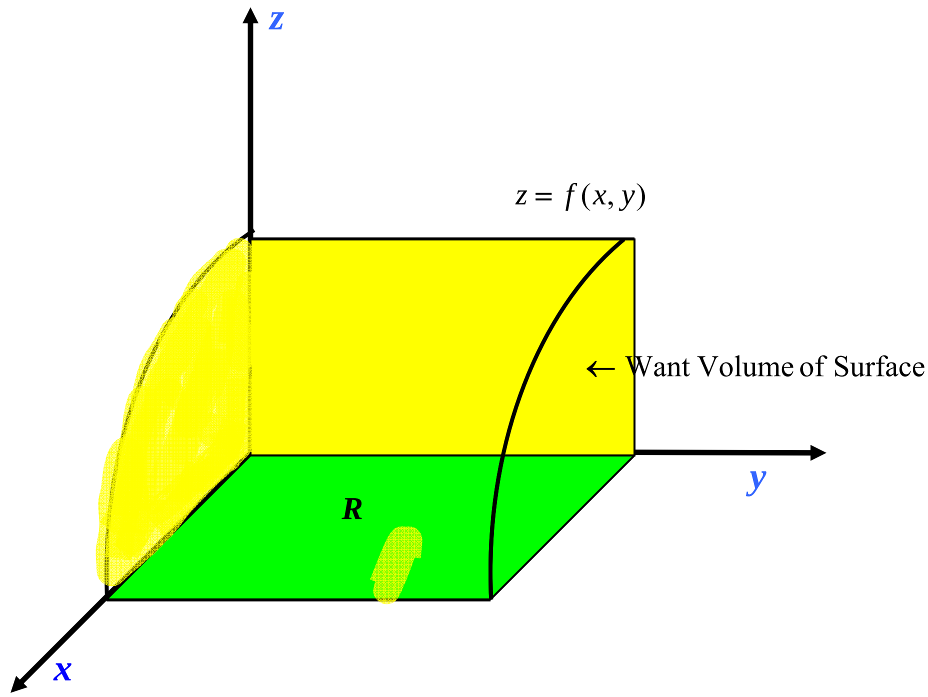
Finding Volume Under a Surface

We want a method for finding the volume between a surface $z = f(x, y)$ and the *x-y* plane. defined by the region *R*.



If $f(x,y) \geq 0$, the volume can be found using a double integral, which is described as follows.

Volume under a Surface

For a function of the two variables $z = f(x,y) \geq 0$ defined over a region *R*, the volume above *R* and under $z = f(x, y)$ is defined by the double integral

$$\text{Volume under R} = \iint_R f(x,y)\,dA$$

Example 9: Find the volume under the surface $z = 2x + y^2$ and above the region bounded by $x = y^2$ and $x = y^3$.

Solution:

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