

Sequences and Series

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Sequences

Throughout we shall use the example $a_n = \frac{3n^2 + (-1)^n n \cos(2n)}{n^2 + 1}$.

First we want to calculate specific members of this sequence. To do so, it is easiest to define the sequence as a function

```
> a := n -> (3*n^2 + (-1)^n*n*cos(2*n))/(n^2+1);
```

$$a := n \rightarrow \frac{3n^2 + (-1)^n n \cos(2n)}{n^2 + 1} \quad (1.1)$$

Computation of say, a_1 or a_{12} , is now very simple.

```
> a(1); a(12);
```

$$\frac{3}{2} - \frac{1}{2} \cos(2)$$
$$\frac{432}{145} + \frac{12}{145} \cos(24) \quad (1.2)$$

and in decimals we have

```
> evalf(a(1)); evalf(a(12));
```

$$1.708073418$$
$$3.014414815 \quad (1.3)$$

Limits of sequences are done the same way they are done for functions, either directly as in

```
> limit( a(n),n=infinity);
```

$$3 \quad (1.4)$$

or in the two-step method (recommended)

```
> Limit( a(n),n=infinity);
value(%);
```

$$\lim_{n \rightarrow \infty} \frac{3n^2 + (-1)^n n \cos(2n)}{n^2 + 1} = 3 \quad (1.5)$$

Of course, the two results are identical. You can also use the template from the *Expressions* button on the left.

```
> lim_{n \to \infty} a(n);
```

$$3 \quad (1.6)$$

Graphical Representation

Sequences are functions which are defined for natural number inputs on, and in our example the $(-1)^n$ term prevents us from plotting the sequence like we would a function. A better way to depict sequences is to plot them as points of the form (n, a_n) . Let's say we want to plot the first twenty points of the sequence. In the first step we write these 20 points as ordered pairs into an array using

[the seq command (output suppressed).

```
> P := seq( [n, a(n)], n=1..20):
```

[For instance, the 12th point in this array is

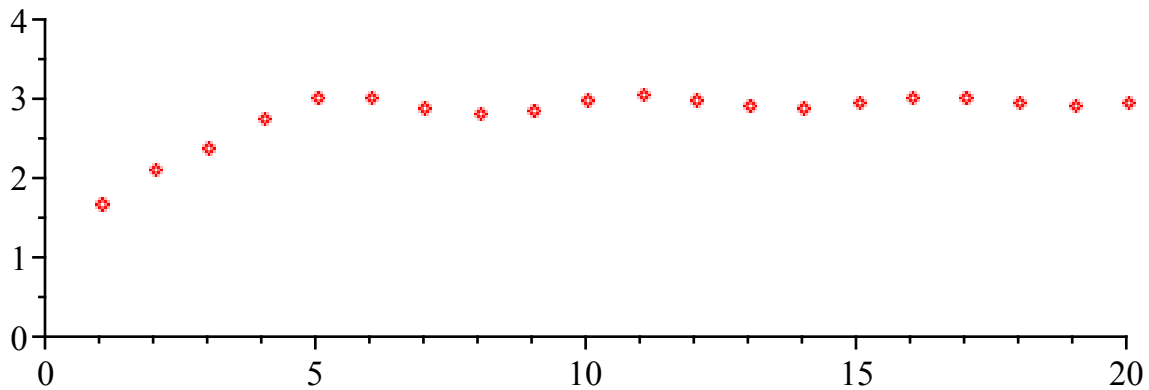
```
> P[12];  
evalf(%)
```

$$\left[12, \frac{432}{145} + \frac{12}{145} \cos(24) \right]$$
$$[12., 3.014414815]$$

(1.7)

[Now we display all twenty points in a graph.

```
> plot( [P], style=point, view = [0..20,0..4]);
```



[The limit of 3 is apparent.

Recursive Sequences

[Here we use the example $a_{n+1} = 2 + \frac{1}{2+a_n}$ with $a_1 = 0$ as initial value.

[First we use a loop to compute the first ten terms of this sequence:

```
> a[1] := 0;  
for n from 1 to 9 do  
a[n+1] := 2 + 1/(2+a[n]);  
end;
```

$$a_1 := 0$$

$$a_2 := \frac{5}{2}$$

$$a_3 := \frac{20}{9}$$

$$a_4 := \frac{85}{38}$$

$$a_5 := \frac{360}{161}$$

$$\begin{aligned}
 a_6 &:= \frac{1525}{682} \\
 a_7 &:= \frac{6460}{2889} \\
 a_8 &:= \frac{27365}{12238} \\
 a_9 &:= \frac{115920}{51841} \\
 a_{10} &:= \frac{491045}{219602}
 \end{aligned}
 \tag{2.1}$$

Now we use the `seq` command to generate the points for the plot

```

> pts := seq( [n,a[n]], n=1..10);
pts := [1, 0], [2,  $\frac{5}{2}$ ], [3,  $\frac{20}{9}$ ], [4,  $\frac{85}{38}$ ], [5,  $\frac{360}{161}$ ], [6,  $\frac{1525}{682}$ ], [7,  $\frac{6460}{2889}$ ], [8,
 $\frac{27365}{12238}$ ], [9,  $\frac{115920}{51841}$ ], [10,  $\frac{491045}{219602}$ ]

```

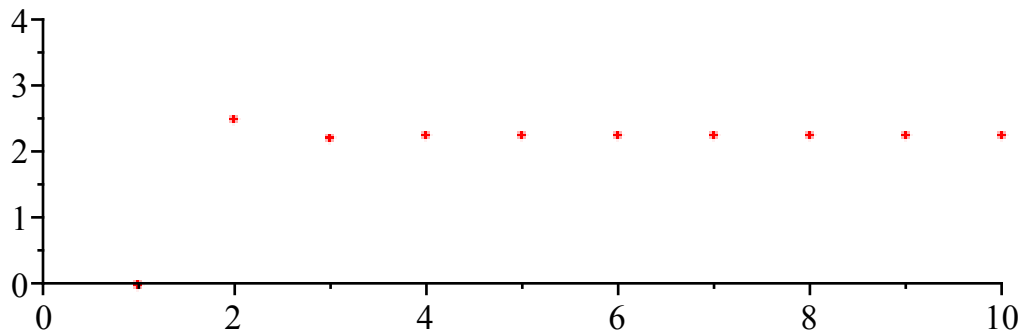
(2.2)

We are now ready to plot the sequence.

```

> plot( [pts], style=point, view=[0..10,0..4]);

```



The graph shows that the sequence converges rather quickly. In order to find the limit, we replace a_{n+1} and a_n by L in the recurrence relation, and then solve the resulting equation for L .

```

> L = 2+1/(2+L);
solve(%,L);

```

$$L = 2 + \frac{1}{2+L}$$

$$\sqrt{5}, -\sqrt{5}$$

(2.3)

Our limit is the positive square root. The computations

```

> evalf( a[8]); evalf(sqrt(5));
2.236067985
2.236067977

```

(2.4)

show that the term a_8 is already very close to the limit.

Finite Sums

```
> restart;
```

We begin by looking at finite sums. These can be computed with the **sum** command, or by using the template on the left.

Example 1: We calculate the first 12 terms in the harmonic series.

```
> sum( 1/n, n=1..12);
```

$$\frac{86021}{27720} \quad (3.1)$$

It is recommended that you use a two step procedure:

```
> Sum( 1/n, n=1..12);  
value(%)
```

$$\sum_{n=1}^{12} \frac{1}{n} = \frac{86021}{27720} \quad (3.2)$$

The numerical value is found with **evalf**

```
> evalf(%);
```

$$3.103210678 \quad (3.3)$$

Using the template instead this calculation becomes

```
>  $\sum_{n=1}^{12} \frac{1}{n};$ 
```

$$\frac{86021}{27720} \quad (3.4)$$

```
> evalf[10]( (3.4) ) # right-click on the fraction and select Approximate
```

$$3.103210678 \quad (3.5)$$

Example 2: Add the first n natural numbers

```
> Sum( k , k=1..n);  
value(%)  
factor(%)
```

$$\sum_{k=1}^n k = \frac{1}{2} (n+1)^2 - \frac{1}{2} n - \frac{1}{2} = \frac{1}{2} n (n+1) \quad (3.6)$$

Example 3: Add the squares of the first n natural numbers

```
> Sum( k^2 , k=1..n);  
value(%)  
factor(%)
```

$$\sum_{k=1}^n k^2 = \frac{1}{3} (n+1)^3 - \frac{1}{2} (n+1)^2 + \frac{1}{6} n + \frac{1}{6} = \frac{1}{6} n (n+1) (2n+1) \quad (3.7)$$

Example 4: Add $1 + 3 + 9 + 27 + \dots + 3^{10}$.

```
> Sum( 3^n, n=0..10);
value(%)
```

$$\sum_{n=0}^{10} 3^n = 88573 \quad (3.8)$$

Series (Infinite Sums)

For infinite sums (series) we use **infinity** as the upper limit of summation.

Example 1: Compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

```
> Sum(1/n^2, n = 1 .. infinity);
value(%)
```

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6} \pi^2 \quad (4.1)$$

Example 2: Evaluate the alternating harmonic series.

```
> Sum( (-1)^(n+1)/n, n=1..infinity);
value(%)
evalf(%)
```

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2) = 0.6931471806 \quad (4.2)$$

Example 3: The series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by the ratio test. The result is surprisingly simple.

```
> Sum( n/2^n , n=1..infinity);
value(%)
```

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

(4.3)

Example 4: This series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3 + 1}$ must converge by the Alternating Series Test. The result is a mess, a decimal approximation is more useful.

```
> Sum( (-1)^(n+1)*n^2/(n^3+1), n=1..infinity);
value(%)
```

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3 + 1}$$

59194845658107141900706407719632896 / ((5 + I√3)³ (7 + I√3)⁴ (9 + I√3)⁴ (-3 + I√3)³ (-5 + I√3)³ (-7 + I√3)⁴ (-9 + I√3)⁴ (11 + I√3)³ (13 + I√3)² (15 + I√3) (-11 + I√3)³ (-15 + I√3) (-13 + I√3)² (3 + I√3)³) + (7796035657138176 (6 + I√3) (16 + 7 I√3) (-228 + I√3) (-2502 + 467 I√3) (2502 + 467 I√3) (228 + I√3) (-16 + 7 I√3) (-6 + I√3) (1 + I√3) (-1 + I√3) ln(2)) / ((3 + I√3)² (5 + I√3)² (9 + I√3)⁴ (-5 + I√3)² (-9 + I√3)⁴ (15 + I√3) (13 + I√3)² (11 + I√3)³ (-15 + I√3) (-13 + I√3)² (-11 + I√3)³ (-7 + I√3)⁴ (-3 + I√3)² (7 + I√3)⁴) + (1352 / 57 I(4379205 + 919631 I√3) (3 + I√3)² (1 + I√3)⁴ (-3 + I√3) √3 (-1 + I√3)⁶ (11 I√3 + 64) (4 + I√3) (2 + I√3) (1/2 Ψ(3/4 - 1/4 I√3) - 1/2 Ψ(1/4 - 1/4 I√3))) / ((7 + I√3)⁴ (9 + I√3)⁴ (-5 + I√3) (-7 + I√3)² (11 + I√3)³ (13 + I√3)² (15 + I√3)) + 4/2223 ((159 + 19 I√3) (29 + 7 I√3) (27 + 37 I√3) (-3 + I√3) (-1 + I√3)³ (3 + I√3) (1 + I√3)⁶ (-2502 + 467 I√3) (237 + 683 I√3) (-60 + 289 I√3) (1/2 Ψ(3/4 + 1/4 I√3) - 1/2 Ψ(1/4 + 1/4 I√3))) / ((5 + I√3) (7 + I√3)² (-7 + I√3)⁴ (-9 + I√3)⁴ (-11 + I√3)³ (-15 + I√3) (-13 + I√3)²)

```
> evalf(%)
```

(4.5)

$$0.2395607471 + 1.10^{-10}i \quad (4.5)$$

It appears that maple had to resort to complex numbers to find a closed form expression for the value of the series. The conversion to decimals left a small contribution from imaginary numbers, which cannot be part of the correct answer, since the original sum does not contain complex numbers.

Example 4: The geometric series formula is built in.

```
> Sum( r^n, n=0..infinity);
value(%);
```

$$\sum_{n=0}^{\infty} r^n = \frac{1}{r-1} \quad (4.6)$$

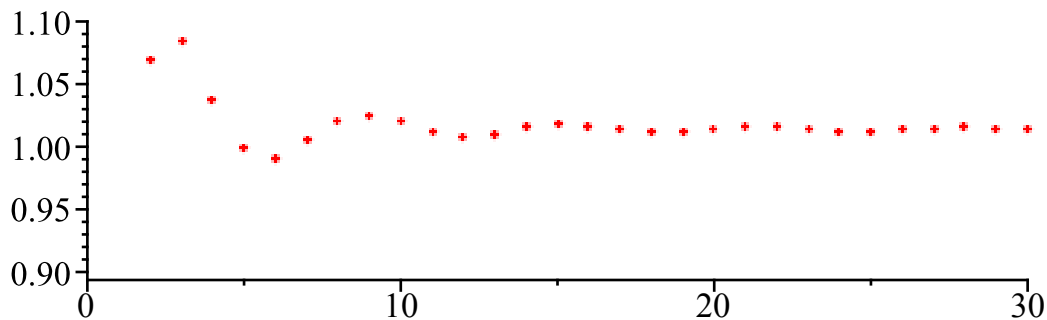
Example 5: The series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges absolutely, but maple can only compute its value using special functions. Decimal approximation shows that the sum is slightly larger than 1.

```
> Sum( sin(n)/n^2, n=1..infinity);
value(%);
evalf(%);
```

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} = -\frac{1}{2} \operatorname{I}(\operatorname{polylog}(2, e^i) - \operatorname{polylog}(2, e^{-i})) = 1.013959132 - 0.1i \quad (4.7)$$

Here is a plot of the first 30 partial sums for this sum.

```
> P := seq([n, sum(sin(k)/k^2, k=1..n)], n=1..30):
plot([P], style=point, view=[0..30, 0.9..1.1]);
```



>