Parametric Curves and Polar Coordinates

Math 251, Fall 2011
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Parametric Curves

We will investigate several aspects of parametric curves in the plane. The curve given by

$$x(t) = \sin(t) \quad \text{and} \quad y(t) = \sin(2\cdot t)$$

will serve as an example throughout, and we begin by defining the curve once and for all.

```
x := t → sin(t); y := t → sin(2·t);
```

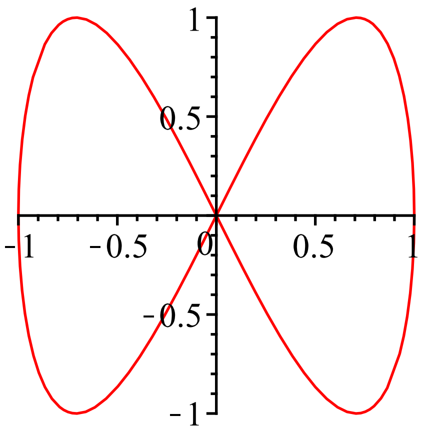
$$t \to \sin(t)$$

$$t \to \sin(2\,t) \tag{1.1}$$

Graphing

A simple modification of the **plot** command lets us graph a parametric curve: Enclose the two curves and the range of the parameter in a single bracket.

```
plot( [x(t), y(t), t = 0..2·Pi], scaling = constrained);
```



The option **scaling=constrained** was added to obtain equal spacing on the coordinate axes.

Slope

Problem: Find the slope of the curve at the point where t=π/3.

Solution: Finding the derivative is easily accomplished using the formula $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

$m := \frac{\mathrm{D}(y)(t)}{\mathrm{D}(x)(t)};$

$$\frac{2\cos(2\,t)}{\cos(t)} \tag{1.2.1}$$

$subs\left(t = \frac{\pi}{3}, m\right);$

$$\frac{2\cos\left(\frac{2}{3}\,\pi\right)}{\cos\left(\frac{1}{3}\,\pi\right)}$$

$simplify(\%);$

$$-2 \tag{1.2.3}$$

Thus the slope at $t = \frac{\pi}{3}$ is -2.

Graph of the Parametric Curve with a Tangent Line

Problem: Graph the curve along with its tangent line at the point where t=π/3 in a common figure.

Solution: The solution requires several steps. First we denote the point which corresponds to t=π/3 by (X,Y), and we compute it as$X := x\left(\frac{\mathrm{Pi}}{3}\right);\ Y := y\left(\frac{\mathrm{Pi}}{3}\right);$

$$\frac{1}{2}\sqrt{3}$$

$$\frac{1}{2}\sqrt{3} \tag{1.3.1}$$

In order to find the slope, we substitute t=π/3 into the formula for m, and call the result M (recall that m was determined above)

$M := subs\left(t = \frac{\mathrm{Pi}}{3}, m\right);$

$$\frac{2\cos\left(\frac{2}{3}\,\pi\right)}{\cos\left(\frac{1}{3}\,\pi\right)} \tag{1.3.2}$$

$M := simplify(M);$

$$-2 \tag{1.3.3}$$

Thus the tangent line has slope m=-2.

We have *several options* to obtain the curve and the tangent line in a common figure. *For one* we could plot the curve and the tangent line separately and then superimpose the two with the **display** command. This approach requires uploading the plots package, and we will not apply this method here. *As an alternative* we can express the tangent line in parametric form and plot the two curves simultaneously, and we will take this route. A **third option** will be shown below.

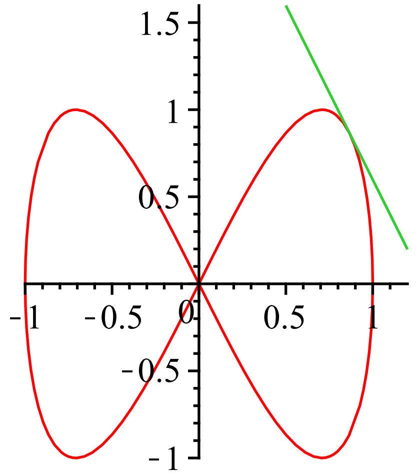
The tangent line y -Y = M(x-X) can be expressed in parametric form by setting x = t, and we get y = Y +M(t-X)

$x = t$

$y = Y + M(t - X)$

This formula can be directly substituted into the plotting command. Result:

$plot(\ [[x(t), y(t), t = 0 ..2\cdot\mathrm{Pi}], [t, Y + M\cdot(t - X), t = 0.5 ..1.2]], scaling = constrained);$



Note that in the plotting command the two parametric curves were enclosed by brackets and that the command has the structure plot([[first curve] , [second curve]], options);

The third option is more elegant. We compute the tangent lines for the two functions x(t) and y(t) at t=π/3 as functions of t, denoted by lx(t) and ly(t), and then we plot (lx(t),ly(t)) as a parametric curve. Here are the details:

$lx := t \rightarrow X + \mathrm{D}(x)\left(\frac{\pi}{3}\right)\cdot\left(t - \frac{\pi}{3}\right);$

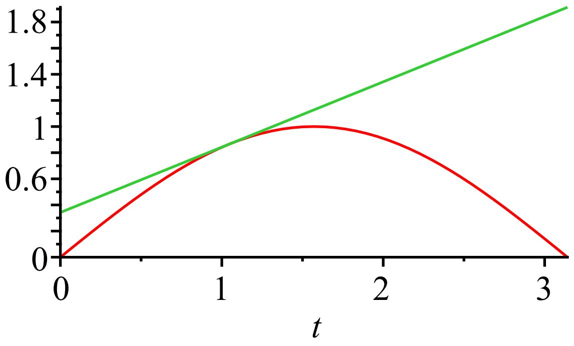
$$t \rightarrow X + \mathrm{D}(x)\left(\frac{1}{3}\,\pi\right)\left(t - \frac{1}{3}\,\pi\right) \tag{1.3.4}$$

$ly := t \rightarrow Y + \mathrm{D}(y)\left(\frac{\pi}{3}\right)\cdot\left(t - \frac{\pi}{3}\right);$

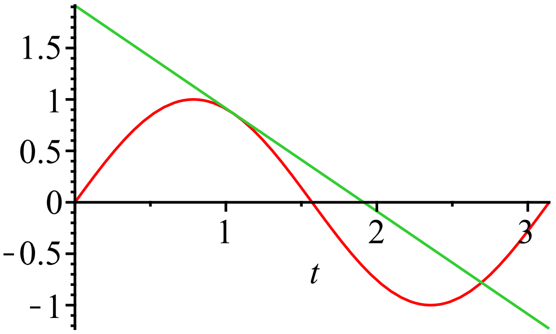
$$t \rightarrow Y + \mathrm{D}(y)\left(\frac{1}{3}\,\pi\right)\left(t - \frac{1}{3}\,\pi\right) \tag{1.3.5}$$

The next two pictures show that we indeed calculated ordinary tangent lines.

$plot([x(t), lx(t)], t = 0 ..\mathrm{Pi});$

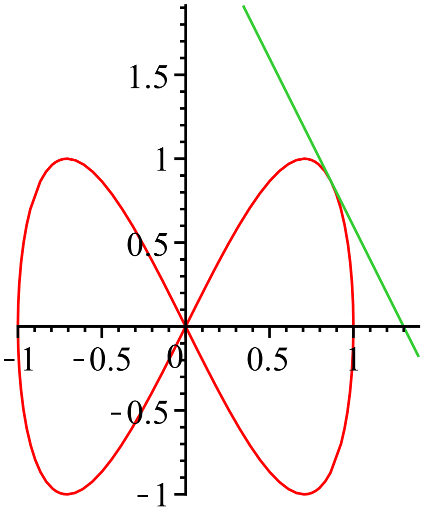


$plot([y(t), ly(t)], t = 0 ..\mathrm{Pi});$



Viewed as parametric curve, the pair lx(t) and ly(t) becomes the tangent line of our curve.

$$plot\left(\left[\left[x(t), y(t), t=0..2\cdot \mathrm{Pi}\right], \left[lx(t), ly(t), t=0..\frac{2\cdot \mathrm{Pi}}{3}\right]\right], scaling=constrained\right);$$



Find Points with a Given Slope

Problem: Find the point in the first quadrant where the slope equals -1.

Solution: We set the slope dy/dx = m, and solve the equation m=-1 for t.

$$m := \frac{\mathrm{D}(y)(t)}{\mathrm{D}(x)(t)};$$

$$\frac{2\cos(2\,t)}{\cos(t)} \tag{1.4.1}$$

$$solve(m=-1, t);$$

$$\arccos\left(-\frac{1}{8}+\frac{1}{8}\sqrt{33}\right), \pi-\arccos\left(\frac{1}{8}+\frac{1}{8}\sqrt{33}\right) \tag{1.4.2}$$

The first value appears to be the desired value for t, and we name it T (copy-and-paste)

$$T := \arccos\left(-\frac{1}{8}+\frac{1}{8}\sqrt{33}\right);$$

$$\arccos\left(-\frac{1}{8}+\frac{1}{8}\sqrt{33}\right) \tag{1.4.3}$$

Then the point with slope m=-1 is located at x(T) and y(T). These points are
$x(T);$

$$\sqrt{1-\left(-\frac{1}{8}+\frac{1}{8}\sqrt{33}\right)^2} \qquad \textbf{(1.4.4)}$$

$y(T);$

$$\sin\left(2\arccos\left(-\frac{1}{8}+\frac{1}{8}\sqrt{33}\right)\right) \qquad \textbf{(1.4.5)}$$

Numerically the coordinates of the point with slope -1 are
$evalf(X(t)); evalf(y(T));$

$$0.8660254040$$

$$0.9550219348 \qquad \textbf{(1.4.6)}$$

Clearly, this point belongs to the first quadrant.

Find the Slopes at the Origin

Problem: What are the slopes as the curve passes through the origin?

Solution: First we need to find the value of the parameter when the curve passes through the origin, that is, we must solve x(t)=0 and y(t)= 0 simultaneously.
$solve(\ \{x(t)=0, y(t)=0\}, t);$

$$\{t=0\}, \{t=\pi\} \qquad \textbf{(1.5.1)}$$

We find two values. Check:
$x(0), y(0); x(\mathrm{Pi}), y(\mathrm{Pi});$

$$0, 0$$

$$0, 0 \qquad \textbf{(1.5.2)}$$

For the slopes we find:
$\frac{\mathrm{D}(y)(0)}{\mathrm{D}(x)(0)};$

$$2 \qquad \textbf{(1.5.3)}$$

$\frac{\mathrm{D}(y)(\pi)}{\mathrm{D}(x)(\pi)};$

$$-2 \qquad \textbf{(1.5.4)}$$

Thus the slopes are +2 and -2.

Arc Length

Problem: What is the arc length of the curve?

Solution: We can implement the arc length formula directly, and we obtain using the templates on the left
$\int_0^{2\pi}\sqrt{\mathrm{D}(x)(t)^2+\mathrm{D}(y)(t)^2}\,\mathrm{d}t$

$$\int_0^{2\pi}\sqrt{\cos(t)^2+4\cos(2t)^2}\,\mathrm{d}t \qquad \textbf{(1.6.1)}$$

$$\xrightarrow{\text{at 5 digits}}$$

$$9.4294$$

Maple couldn't find an analytic solution, and we accept a numerical approximation in its place.

Area

Problem: Find the area enclosed by the curve.

Solution: The area between a parametric curve and the x-axis can be computed as follows:

$$A = \int_a^a y\,\mathrm{d}x \;=\; \int_{t_1}^{t_2} y(t)\cdot x'(t)\;\mathrm{d}t$$

where t_1 and t_2 are to be selected to match the endpoints.
In our example we use symmetry, and multiply the area in the first quadrant by 4 to get the total area. The parameter interval is [0,π/2], and the area becomes

$$4\cdot\int_0^{\frac{\pi}{2}} y(t)\cdot \mathrm{D}(x)(t)\;\mathrm{d}t$$

$$\frac{8}{3} \tag{1.7.1}$$

Polar Coordinates

restart;
Here we use the example r(θ) = 3 + 5 sin θ, and we shall work several problems related to this curve. First, we define r once and for all
$r := t \to 3 + 5\cdot\sin(t);$

$$t \to 3 + 5\sin(t) \tag{2.1}$$

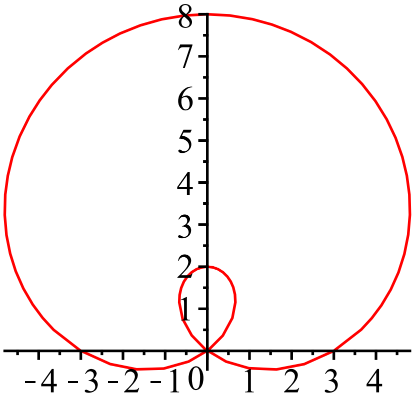
It is easier to work with t rather than typing θ all the time. However, since r is a function (change the input from t to θ) we have
$r(\theta);$

$$3 + 5\sin(\theta) \tag{2.2}$$

Polar Graphs

Plotting in polar coordinates requires the option **coords=polar,** everything else is pretty much the same. This is the equivalent of setting your calculator to the polar function mode. Here is the graph:

plot(*r*, 0 ..2·Pi, *coords* = polar, *scaling* = *constrained*);

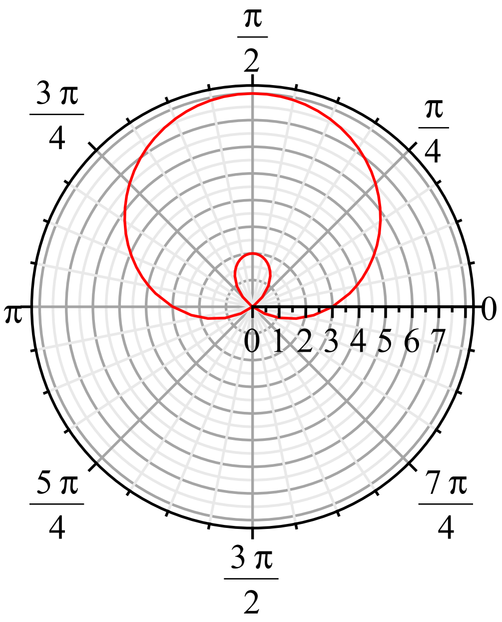


The plots package offers a nice alternative. In order to use it, we need to activate it first with the command

```
with(plots) :
```

Use a colon to avoid unnecessary output. Now we are ready to plot.

```
polarplot(r(t), t=0..2·π);
```



Slope at a Given Point

Problem: What is the slope of the curve at the point (x,y)=(3,0)?

Solution: First of all we realize that the point (3,0) is obtained by setting θ = 0. For the slope dy/dx we follow the usual procedure for parametric curves, this time with the **diff** command

$$\frac{diff(r(t)\cdot\sin(t), t)}{diff(r(t)\cdot\cos(t), t)}$$

$$\frac{5\cos(t)\sin(t) + (3+5\sin(t))\cos(t)}{5\cos(t)^2 - (3+5\sin(t))\sin(t)} \tag{2.2.1}$$

$simplify(\%);$

$$\frac{\cos(t)\ (10\sin(t)+3)}{10\cos(t)^2-3\sin(t)-5} \tag{2.2.2}$$

And now we set t = 0

$subs(\ t=0,\ \%);$

$$\frac{\cos(0)\ (10\sin(0)+3)}{10\cos(0)^2-3\sin(0)-5} \tag{2.2.3}$$

$simplify(\%);$

$$\frac{3}{5} \tag{2.2.4}$$

Thus the desired slope is 3/5.

Graph of the Polar Curve with a Tangent Line

Problem: Graph the curve along with its tangent line at the point (3,0) in a common figure.

Solution: We know from the last problem that the slope is 3/5, and using the point-slope form of a line, the tangent line is given by

$$y=\frac{3}{5}(x-3)$$

For the graph we have two alternatives: Either express the line in polar coordinates and use a single ploting statement, or graph the two pieces separately and superimpose the graphs. We will demonstrate both.

In polar coordinates the line becomes

$r\cdot\sin(t) = \frac{3}{5}\cdot(r\cdot\cos(t)-3);$

$$r\sin(t)=\frac{3}{5}\,r\cos(t)-\frac{9}{5} \tag{2.3.1}$$

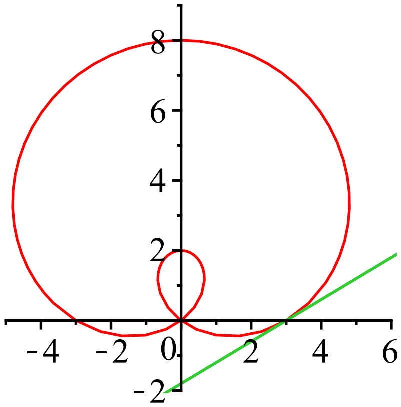
and solving this equation for r we find

$solve(\%,\ r);$

$$\frac{9}{-5\sin(t)+3\cos(t)} \tag{2.3.2}$$

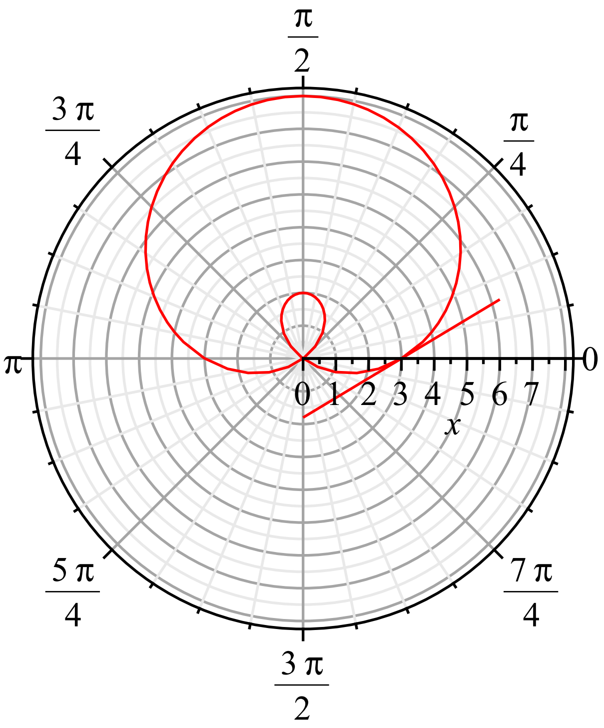
This is an expression for the tangent line in polar coordinates, and with a simple copy-and-paste we obtain

$plot\left(\left[r(t),\ \frac{9}{-5\sin(t)+3\cos(t)}\right],\ \mathrm{t}=-\mathrm{Pi}\,..\mathrm{Pi},\ \mathrm{coords}=\mathrm{polar},\ \mathrm{view}=[-5\,..6,-2\,..9],\ scaling=constrained\right);$



The alternative requires the plotting package (which is already activated).

$A := polarplot(\ r, 0..2\cdot\mathrm{Pi})$: # *store the graph of the curve as A*
$B := plot\left(\frac{3}{5}\cdot(x-3), x=0..6\right)$: # *savethe tangent line in rectangular coordinates as B*
$display(A, B)$; # *superimpose*



Slopes at the Origin

Problem: What are the slopes of the curve as it passes though the origin?

Solution: First we set r=0 to find the points where the curve passes through the origin.
$solve(\ r(t)=0, t)$;

$$-\arcsin\left(\frac{3}{5}\right) \tag{2.4.1}$$

This result is of limited use. But applying our knowledge of trigonometry we conclude that r(t)=0 for

t = 2π - arcsin(3/5)   and   t = π + arcsin(3/5)

Let's confirm this:

$T1 := \pi + \arcsin\left(\frac{3}{5}\right);\ r(T1);$

$$\pi + \arcsin\left(\frac{3}{5}\right)$$

$$0 \tag{2.4.2}$$

$T2 := 2 \cdot \pi - \arcsin\left(\frac{3}{5}\right);\ r(T2);$

$$2\,\pi - \arcsin\left(\frac{3}{5}\right)$$

$$0 \tag{2.4.3}$$

Now we calculate the slope at these points.

$\frac{diff(\,r(t)\cdot\sin(t),\,t)}{diff(\,r(t)\cdot\cos(t),\,t)};$

$$\frac{5\cos(t)\sin(t) + (3 + 5\sin(t))\cos(t)}{5\cos(t)^2 - (3 + 5\sin(t))\sin(t)} \tag{2.4.4}$$

$subs(t = T1, \%);$

$$\left(5\cos\left(\pi + \arcsin\left(\frac{3}{5}\right)\right)\sin\left(\pi + \arcsin\left(\frac{3}{5}\right)\right) + \left(3 + 5\sin\left(\pi + \arcsin\left(\frac{3}{5}\right)\right)\right)\cos\left(\pi + \arcsin\left(\frac{3}{5}\right)\right)\right) \Big/ \left(5\cos\left(\pi + \arcsin\left(\frac{3}{5}\right)\right)^2 - \left(3 + 5\sin\left(\pi + \arcsin\left(\frac{3}{5}\right)\right)\right)\sin\left(\pi + \arcsin\left(\frac{3}{5}\right)\right)\right) \tag{2.4.5}$$

$simplify(\%);$

$$\frac{3}{4} \tag{2.4.6}$$

$\frac{diff(\,r(t)\cdot\sin(t),\,t)}{diff(\,r(t)\cdot\cos(t),\,t)};$

$$\frac{5\cos(t)\sin(t) + (3 + 5\sin(t))\cos(t)}{5\cos(t)^2 - (3 + 5\sin(t))\sin(t)} \tag{2.4.7}$$

$simplify(\,subs(t = T2, \%)\,);$

$$-\frac{3}{4} \tag{2.4.8}$$

There is a Much easier way to determine the slope. We know from class that the slope is the tangent of the respective angles.

$\tan(T1);\ \tan(T2);$

$$\frac{3}{4}$$

$$-\frac{3}{4} \tag{2.4.9}$$

Arc Length

Problem: Find the arc length of the curve.

Solution: Here we implement the familiar formula for the arc length of a polar curve directly, and let maple do the work.

$$\int_0^{2\cdot\pi} \sqrt{r(t)^2 + \mathrm{D}(r)(t)^2}\, \mathrm{d}t$$

$$48\,\mathrm{EllipticE}\left(\frac{1}{4}\sqrt{15}\right) - \mathrm{EllipticPi}\left(\frac{4}{17}\sqrt{17}, \frac{15}{16}, \frac{1}{4}\sqrt{15}\right) - 16\,\mathrm{EllipticE}\left(\frac{1}{2}\sqrt{2}, \frac{1}{4}\sqrt{15}\right) \qquad \textbf{(2.5.1)}$$

$$\xrightarrow{\text{at 10 digits}}$$

$$34.31368713 \qquad \textbf{(2.5.2)}$$

The result involves special functions, but numerical evaluation leads to an acceptable result.

Area

Problem: Compute the area enclosed by the inner loop of the curve.

Solution: The inner part of the curve is traced when T1 < t < T2 (the numbers T1 and T2 were found above). Using the area formula for polar curves we find that

$$\frac{1}{2}\cdot\int_{T1}^{T2} r(t)^2\, \mathrm{d}t$$

$$\frac{43}{4}\pi - \frac{43}{2}\arcsin\left(\frac{3}{5}\right) - 18 \qquad \textbf{(2.6.1)}$$

evalf(%);

$$1.93684719 \qquad \textbf{(2.6.2)}$$

The area is roughly two square units.