

Limits in Maple

In this worksheet we shall compute limits with maple. We begin by examining basic strategies, go over one-sided limits, look at limits at infinity, and finally use limits to look at the limit definition of derivatives.

Basics

Throughout most of this section we use the example $\lim_{x \rightarrow 1} \frac{\ln(x^2)}{x-1}$.

1. Direct Method.

The basic command for taking limits in maple is **limit**. In our example we would enter

$$\text{limit}\left(\frac{\ln(x^2)}{x-1}, x=1\right);$$

2 (1.1)

and maple returns the limit.

2. Inert Method.

If we use the command with an upper case L, we obtain

$$\text{Limit}\left(\frac{\ln(x^2)}{x-1}, x=1\right);$$
$$\lim_{x \rightarrow 1} \frac{\ln(x^2)}{x-1}$$

(1.2)

Here our limit problem is displayed, but nothing is calculated. If we follow up with the **value** command, we obtain the desired result.

$$\text{value}(\%);$$

2 (1.3)

3. Expressions Template

The *Expressions* listing on the right contains the limit command. Select it, and use the Tab-key to move through the parameters, and obtain

$$\lim_{x \rightarrow 1} \frac{\ln(x^2)}{x-1}$$

2 (1.4)

4. Define the Function or Expression First

If the function involved is complicated, or if the same function is used many times, it is useful to define it first, and then apply the limit command. In our case we would do the following:

$$f := x \rightarrow \frac{\ln(x^2)}{x-1};$$
$$x \rightarrow \frac{\ln(x^2)}{x-1}$$

(1.5)

$$\text{limit}(f(x), x=1);$$

2 (1.6)

More Examples: These are the three special limits on page 54 of our text.

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x};$$

1 (1.7)

$$\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x}$$

0 (1.8)

$$\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}$$

e (1.9)

Maple cannot perform miracles and find limits that do not exist:

$$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right);$$

-1..1 (1.10)

$$\lim_{x \rightarrow 0} \frac{|x|}{x};$$

undefined (1.11)

One-Sided Limits

For one-sided limits we need to add the direction from which the limit is taken.

Example

Let's take the piecewise defined function $f(x) = \begin{cases} x^2 - 3 & x < 2 \\ x + 1 & x \geq 2 \end{cases}$ and have a look at its graph.

$$f := x \rightarrow \begin{cases} x^2 - 3 & x < 2 \\ x + 1 & x \geq 2 \end{cases};$$

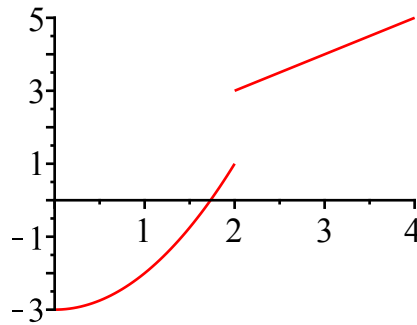
$x \rightarrow \text{piecewise}(x < 2, x^2 - 3, 2 \leq x, x + 1)$ (2.1)

$f(x)$; # yes, it's there

$$\begin{cases} x^2 - 3 & x < 2 \\ 1 + x & 2 \leq x \end{cases}$$

(2.2)

$\text{plot}(f, 0..4, \text{discont} = \text{true});$



Clearly, we have different limits as we approach 2 from either side. Let's find them
 $\lim_{x \rightarrow 2^-} f(x)$;

$$1 \tag{2.3}$$

$\lim_{x \rightarrow 2^+} f(x)$;

$$3 \tag{2.4}$$

and the limit as such does not exist

$\lim_{x \rightarrow 2} f(x)$;

$$\text{undefined} \tag{2.5}$$

More Examples:

First we look at $\frac{|x|}{x}$ as x approaches zero.

$\lim_{x \rightarrow 0^-} \left(\frac{|x|}{x} \right)$;

$$-1 \tag{2.6}$$

$\lim_{x \rightarrow 0^+} \left(\frac{|x|}{x} \right)$;

$$1 \tag{2.7}$$

or, the same in clickable math,

$\lim_{x \rightarrow 0^+} \frac{|x|}{x}$;

$$1 \tag{2.8}$$

$\lim_{x \rightarrow 0^-} \frac{|x|}{x}$

$$-1 \tag{2.9}$$

If a limit exists, the one-sided derivatives will be identical and have that value. For instance take

$f(x) = x^2$ at $x=3$.

$f := x \rightarrow x^2$;

$$x \rightarrow x^2 \tag{2.10}$$

$\lim_{x \rightarrow 3^-} f(x)$;

$$9 \tag{2.11}$$

$\lim_{x \rightarrow 3^+} f(x)$;

$$9 \tag{2.12}$$

$\lim_{x \rightarrow 3} f(x)$;

$$9 \tag{2.13}$$

Infinity in Limits

If you want to compute limits where x approaches infinity, just enter **infinity** or use the ∞ symbol from the *Common Symbols* list on the left.

Examples:

$\text{limit}(\arctan(x), x = \text{infinity});$

$$\frac{1}{2} \pi \quad (3.1)$$

or, equivalently

$\lim_{x \rightarrow \infty} \arctan(x);$

$$\frac{1}{2} \pi \quad (3.2)$$

If a limit does not exist because the function grows without a bound, maple will give ∞ as a result.

Examples:

$\text{limit}\left(\ln\left(\frac{1}{x}\right), x = 0\right);$

$$\infty \quad (3.3)$$

$\text{limit}\left(\frac{x^3 - 6x + 8}{x^2 + 5}, x = \text{infinity}\right);$

$$\infty \quad (3.4)$$

$\lim_{x \rightarrow \frac{\pi}{2}} \tan(x);$

$$\text{undefined} \quad (3.5)$$

This limit is undefined, since, depending on the direction, the tangent approaches $-\infty$ or $+\infty$. With one-sided limits we get

$\text{limit}\left(\tan(x), x = \frac{\pi}{2}, \text{left}\right);$

$$\infty \quad (3.6)$$

$\text{limit}\left(\tan(x), x = \frac{\pi}{2}, \text{right}\right);$

$$-\infty \quad (3.7)$$

Limit Definition of the Derivative

As you know, the derivative is defined as the limit $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, and once the function $f(x)$ is defined, we can use this command to compute the derivative.

Example: Compute the derivative of the function $f(x) = x^2 \sin(x)$.

First we define the function

$f := x \rightarrow x^2 \cdot \sin(x);$

$$x \rightarrow x^2 \sin(x) \quad (4.1)$$

Now compute the limit

$$\lim_{h \rightarrow 0} \frac{(f(x+h) - f(x))}{h};$$

$$2 \sin(x) x + \cos(x) x^2 \quad (4.2)$$

Example: Compute the derivative of $f(x) = \frac{x^2}{x^2 + 1}$.

Occasionally it is useful to break the problem down into several steps, and we will do it here.

$$f := x \rightarrow \frac{x^2}{(x^2 + 1)};$$

$$x \rightarrow \frac{x^2}{x^2 + 1} \quad (4.3)$$

$$\frac{(f(x+h) - f(x))}{h};$$

$$\frac{\frac{(x+h)^2}{(x+h)^2 + 1} - \frac{x^2}{x^2 + 1}}{h} \quad (4.4)$$

simplify(%);

$$\frac{2x + h}{(x^2 + 2xh + h^2 + 1)(x^2 + 1)} \quad (4.5)$$

limit(% , h=0);

$$\frac{2x}{(x^2 + 1)^2} \quad (4.6)$$

Of course, we can also go for the derivative directly, but this hides all the steps which went into the result.

$$\lim_{h \rightarrow 0} \frac{(f(x+h) - f(x))}{h};$$

$$\frac{2x}{(x^2 + 1)^2} \quad (4.7)$$

We can also compute the derivative using the alternate definition $f'(x) = \lim_{u \rightarrow x} \frac{f(u) - f(x)}{u - x}$.

Examples revisited:

$$f := x \rightarrow x^2 \cdot \sin(x);$$

$$x \rightarrow x^2 \sin(x) \quad (4.8)$$

$$\lim_{u \rightarrow x} \frac{f(u) - f(x)}{u - x};$$

$$2 \sin(x) x + \cos(x) x^2 \quad (4.9)$$

$$f := x \rightarrow \frac{x^2}{x^2 + 1};$$

$$x \rightarrow \frac{x^2}{x^2 + 1} \quad (4.10)$$

$f(u) - f(x);$

$$\frac{u^2}{u^2 + 1} - \frac{x^2}{x^2 + 1} \quad (4.11)$$

simplify(%);

$$\frac{u^2 - x^2}{(u^2 + 1)(x^2 + 1)} \quad (4.12)$$

$\frac{\%}{u - x};$

$$\frac{u^2 - x^2}{(u^2 + 1)(x^2 + 1)(u - x)} \quad (4.13)$$

simplify(%);

$$\frac{u + x}{(u^2 + 1)(x^2 + 1)} \quad (4.14)$$

limit(% , u = x);

$$\frac{2x}{(x^2 + 1)^2} \quad (4.15)$$

Or directly, as

$\lim_{u \rightarrow x} \frac{f(u) - f(x)}{u - x}$

$$\frac{2x}{(x^2 + 1)^2} \quad (4.16)$$