4. The product of two positive odd integers is always odd.

TEST HYPOTHESIS:

$x = 3$	$y = 5$	$x \cdot y = 3 \cdot 5 = 15$ ✓
$x = 5$	$y = 7$	$x \cdot y = 5 \cdot 7 = 35$ ✓
$x = 9$	$y = 11$	$x \cdot y = 9 \cdot 11 = 99$ ✓

PROOF: LET X BE AN ODD INTEGER. FROM THE DEFINITION OF ODD, THERE EXISTS AN INTEGER K SUCH $X = 2K+1$. LET Y BE AN ODD INTEGER. FROM THE DEFINITION OF ODD THERE EXISTS AN INTEGER L SUCH THAT $Y = 2L+1$.

$$X \cdot Y = (2K+1)(2L+1) \quad \text{SUBSTITUTION}$$

$$= 4KL + 2K + L + 2L + 1$$

$$= 4KL + 2K + 2L + 1$$

$$= 2(2KL + K + L) + 1$$

SINCE K & L WERE INTEGERS, $2KL + K + L$ IS ALSO AN INTEGER. FROM THE DEFINITION OF ODD, $2 \cdot \text{INTEGER} + 1$ IS ODD. SO, $X \cdot Y$ IS ODD. ■

5. The product of an odd integer and an even integer is always even.

TEST HYPOTHESIS: $X = 5 \quad Y = 4 \quad X \cdot Y = 20$ ✓

$X = 7 \quad Y = 6 \quad X \cdot Y = 42$ ✓

$X = 11 \quad Y = 4 \quad X \cdot Y = 44$ ✓

PROOF: LET X BE AN ODD INTEGER. FROM THE DEFINITION OF ODD, THERE EXISTS AN INTEGER K, SUCH THAT $X = 2K + 1$.
LET Y BE AN EVEN INTEGER. FROM THE DEFINITION OF EVEN THERE EXISTS AN INTEGER L SUCH THAT $Y = 2L$.

$$X \cdot Y = (2K+1)(2L) \quad \text{(SUBSTITUTION)}$$
$$= 4KL + 2L$$
$$= 2(2KL + L)$$

SINCE K & L ARE INTEGERS, $2KL + L$ IS ALSO AN INTEGER. FROM THE DEFINITION OF EVEN, $2 \cdot$ INTEGER IS EVEN. $X \cdot Y = 2 \cdot$ INTEGER SO $X \cdot Y$ IS EVEN. ∎

6. If one integer is even and another is divisible by 3, then their product is divisible by 6.

TEST HYPOTHESIS:

$X = 6 \quad Y = 9 \quad X \cdot Y = 6 \cdot 9 = 54$ ✓

$X = 10 \quad Y = 12 \quad X \cdot Y = 10 \cdot 12 = 120$ ✓

$X = 2 \quad Y = 24 \quad X \cdot Y = 2 \cdot 24 = 48$ ✓

PROOF: LET X BE AN EVEN INTEGER. FROM THE DEFINITION OF EVEN, THERE EXISTS AN INTEGER K SUCH THAT $X = 2K$. LET Y BE AN INTEGER DIVISIBLE BY 3. FROM THE DEFINITION OF DIVISIBLE BY 3, THERE EXISTS AN INTEGER L SUCH THAT $Y = 3L$.

$$
\begin{aligned}
X \cdot Y &= (2K)(3L) \quad \text{SUBSTITUTION} \\
&= 6KL \\
&= 6(KL)
\end{aligned}
$$

SINCE K & L ARE INTEGERS, $K \cdot L$ IS ALSO AN INTEGER. FROM THE DEFINITION OF DIVISIBLE BY 6, $6 \cdot$ INTEGER IS DIVISIBLE BY 6. $X \cdot Y = 6 \cdot$ INTEGER, SO $X \cdot Y$ IS DIVISIBLE BY 6. ∎