Chapter 2: Writing Proofs About Numbers

Section 2.1 and 2.2: Introduction to Proofs

Examples of claims and proving claims for specific cases.

1. If n is odd, then $n^2 + 4$ is a prime number for n ≥ 1.

IS THIS TRUE?

$n = 1 \qquad 1^2 + 4 = 1 + 4 = 5$ ✓

$n = 5 \qquad 5^2 + 4 = 25 + 4 = 29$ ✓

$n = 3 \qquad 3^2 + 4 = 9 + 4 = 13$ ✓

2. If n is odd, then $n^3 - n$ is evenly divisible by 4 for n ≥ 1.

IS THIS TRUE?

$n = 1 \qquad 1^3 - 1 = 1 - 1 = 0$ ✓

$n = 3 \qquad 3^3 - 3 = 27 - 3 = 24$ ✓

$n = 5 \qquad 5^3 - 5 = 125 - 5 = 120$ ✓

If all specific cases are true, then a proof for the general case can be written. If a counterexample is found for a claim, the claim cannot be proven.

To prove a statement for the general case

1. Pick values for n to test the hypothesis.
2. If all cases for n are true, a proof for the general case can be written.

Basic definitions used for proofs (part I)

1. An integer n is **EVEN** IF......

   IT CAN BE WRITTEN IN THE FORM $n = 2K$ FOR SOME INTEGER K.

2. An integer n is **ODD** IF......

   IT CAN BE WRITTEN IN THE FORM $n = 2K+1$ FOR SOME INTEGER K.

3. An integer n is **DIVISIBLE BY 4** IF......

   IT CAN BE WRITTEN IN THE FORM $4K$ FOR SOME INTEGER K.

Prove the following (if possible).

1. The sum of an odd integer with an even integer is odd.

TEST HYPOTHESIS: $x=3$ $y=2$ $3+2=5$ ✓

$x=7$ $y=4$ $7+4=11$ ✓

$x=9$ $y=10$ $9+10=19$ ✓

PROOF: LET x BE AN ODD INTEGER. FROM THE DEFINITION OF ODD $x=2k+1$ FOR SOME INTEGER k. LET y BE AN EVEN INTEGER. FROM THE DEFINITION OF EVEN, $y=2L$ FOR SOME INTEGER L.

$$x+y=(2k+1)+2L \quad \text{(SUBSTITUTION)}$$
$$=2k+2L+1$$
$$=2(k+L)+1$$

SINCE k & L ARE INTEGERS, THEN $(k+L)$ IS ALSO AN INTEGER. FROM THE DEFINITION OF ODD, $2\cdot\text{INTEGER}+1$ IS ODD. THEREFORE, $x+y$ IS ODD. ∎

2. If n is even, then n^2 is divisible by 4.

TEST HYPOTHESIS: $n = 2 \quad n^2 = 2^2 = 4$ ✓

$n = 6 \quad n^2 = 6^2 = 36$ ✓

$n = 10 \quad n^2 = 10^2 = 100$ ✓

PROOF: LET n BE AN EVEN INTEGER. FROM THE DEFINITION OF EVEN, $n = 2K$ FOR AN INTEGER K.

$$n^2 = (2K)^2 \quad \text{(SUBSTITUTION)}$$

$$= 4K^2$$

SINCE K IS AN INTEGER, K^2 IS ALSO AN INTEGER. FROM THE DEFINITION OF DIVISIBLE BY 4, $4 \cdot \text{INTEGER}$ IS DIVISIBLE BY 4.

THEREFORE n^2 IS DIVISIBLE BY 4. ■

3. The sum of two positive odd integers is always even.

TEST HYPOTHESIS:

$x = 7$	$y = 7$	$x + y = 7 + 7 = 14$ ✓
$x = 5$	$y = 9$	$x + y = 5 + 9 = 14$ ✓
$x = 3$	$y = 13$	$x + y = 3 + 13 = 16$ ✓

PROOF: LET x BE AN ODD INTEGER. FROM THE DEFINITION OF ODD, $x = 2k+1$ FOR SOME INTEGER k. LET y BE AN ODD INTEGER. FROM THE DEFINITION OF ODD, $y = 2L+1$ FOR SOME INTEGER L.

$$x + y = (2k+1) + (2L+1) \quad \text{SUBSTITUTION}$$
$$= 2k + 2L + 2$$
$$= 2(k + L + 1)$$

SINCE k & L ARE INTEGERS, $k + L + 1$ IS ALSO AN INTEGER. FROM THE DEFINITION OF EVEN, $2 \cdot$ INTEGER IS EVEN. SO $x + y$ IS EVEN. ■