3. Prove that if n is odd, the sum of itself and its square is even.

Test Hypothesis:

DIRECT PROOF

4. Prove using an ***indirect*** proof, the following:

If $m + n$ is odd, then m or n must be even.

P → Q

$\sim Q \longrightarrow \sim P$

"IF m AND n ARE ODD, THEN m+n IS EVEN"

Test Hypothesis:

5. For any integer n, $n - 3n$ is always even. **(HINT: Use both cases for n, when n is even and when n is odd).**

Test Hypothesis: $n = 4 \quad n - 3n = 4 - 3(4) = 4 - 12 = -8$ ✓

$n = 1 \quad n - 3n = 1 - 3(1) = 1 - 3 = -2$ ✓

$n = 5 \quad n - 3n = 5 - 3(5) = 5 - 15 = -10$ ✓

PROOF:

- CASE 1: LET n BE AN EVEN INTEGER. FROM THE DEFINITION OF EVEN, THERE EXISTS AN INTEGER K SUCH THAT $n = 2K$. +1

$$n - 3n = 2K - 3(2K) \quad \text{[SUBSTITUTION]}$$
$$= 2K - 6K$$
$$= 2(K - 3K) \quad +1$$

SINCE K IS AN INTEGER, $K - 3K$ IS ALSO AN INTEGER. FROM THE DEFINITION OF EVEN, $2 \cdot$ INTEGER IS EVEN. $n - 3n = 2 \cdot$ INTEGER AND IS THEREFORE EVEN.

- CASE 2: LET n BE AN ODD INTEGER. FROM THE DEFINITION OF ODD, THERE EXISTS AN INTEGER L SUCH THAT $n = 2L+1$. +1

$$n - 3n = 2L+1 - 3(2L+1)$$
$$= 2L+1 - 6L-3$$
$$= -4L - 2$$
$$= 2(-2L-1) \quad +1$$

SINCE L IS AN INTEGER, $-2L-1$ IS ALSO AN INTEGER. FROM THE DEFINITION OF EVEN, $2 \cdot$ INTEGER IS EVEN. $n - 3n = 2 \cdot$ INTEGER AND IS THEREFORE EVEN.

SINCE BOTH CASES FOR n WERE PROVEN TRUE, THEN $n - 3n$ IS ALWAYS EVEN FOR ANY INTEGER n. ■ +1

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GOAL: $\left[\frac{(k+1)(k+2)}{2}\right]^2$

6. Prove using induction that $1^3+2^3+3^3+.......+n^3=\left[\frac{n(n+1)}{2}\right]^2$

Test Hypothesis: $n=3,\ P(3): 1+8+27=36$ ✓ FORMULA: $\left[\frac{3(3+1)}{2}\right]^2=\left[\frac{12}{2}\right]^2=6^2=36$

$n=4, P(4)=1+8+27+64=100$ ✓ FORMULA: $\left[\frac{4(4+1)}{2}\right]^2=\left[\frac{20}{2}\right]^2=10^2=100$ ✓

$n=5, P(5)=1+8+27+64+125=225$ ✓ FORMULA: $\left[\frac{5(5+1)}{2}\right]^2=\left[\frac{30}{2}\right]^2=15^2=225$

Proof by Induction:

+1

$n=1,\ P(1)=1$ ✓ FORMULA: $\left[\frac{1(1+1)}{2}\right]^2=\left[\frac{2}{2}\right]^2=1^2=1$ ✓

$n=2,\ P(2)=1+8=9$ FORMULA: $\left[\frac{2(2+1)}{2}\right]^2=\left[\frac{6}{2}\right]^2=3^2=9$ ✓

$\vdots$

$n=k,\ P(k)=1^3+2^3+3^3+\ldots\ldots k^3=\left[\frac{k(k+1)}{2}\right]^2$ ✓

+1 $n=k+1,\ P(k+1)=1^3+2^3+3^3+\ldots\ldots k^3+(k+1)^3=\left[\frac{k(k+1)}{2}\right]^2+(k+1)^3$

$$=\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$=\frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4}$$

+2

$$=\frac{k^2\cancel{(k+1)^2}}{4}+\frac{4\cancel{(k+1)^2}(k+1)}{4}$$

$$=(k+1)^2\left[\frac{k^2}{4}+\frac{4(k+1)}{4}\right]$$

$$=(k+1)^2\left[\frac{k^2+4(k+1)}{4}\right]$$

$$=(k+1)^2\left[\frac{k^2+4k+4}{4}\right] \to (k+2)(k+2)$$

$$=(k+1)^2\,\frac{(k+2)^2}{4}$$

$$=\left[\frac{(k+1)(k+2)}{2}\right]^2$$

+1 P(1) WAS TRUE, P(k) WAS TRUE, P(k+1) WAS TRUE, THEREFORE

$1^3+2^3+3^3+\ldots\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2$. ■

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