Section 2.2: Indirect proofs

An indirect proof is a proof created by using the contrapositive of a given conditional statement.

Recall, $P \rightarrow Q \equiv \sim Q \rightarrow \sim P$

Create the contrapositive for the following statements:

If $\underbrace{\text{it is sunny outside}}_{P}$, then $\underbrace{\text{we will go skiing}}_{Q}$.

$P \longrightarrow Q$

IF WE DO NOT GO SKIING, THEN IT IS NOT SUNNY OUTSIDE.

If $\underbrace{\text{n is even}}_{P}$, the $\underbrace{\text{2n is odd}}_{Q}$.

$P \longrightarrow Q$

$\sim Q \longrightarrow \sim P$

IF 2n IS EVEN, THEN n IS ODD.

Examples of indirect proofs.

1. Prove using an indirect proof the following statement.

$$\underbrace{\text{If 3n Is odd}}_{P}\text{, then } \underbrace{\text{n is odd}}_{Q}. \qquad P \to Q$$

$$P \longrightarrow Q$$

$$\sim Q \longrightarrow \sim P$$

" IF n IS EVEN, THEN $3n$ IS EVEN."

TEST HYPOTHESIS: $n = 4$ $\quad 3n = 3(4) = 12$ ✓

$n = 10$ $\quad 3n = 3(10) = 30$ ✓

$n = 20$ $\quad 3n = 3(20) = 60$ ✓

PROOF: LET n BE AN EVEN INTEGER. FROM THE DEFINITION OF EVEN, THERE EXISTS AN INTEGER k SUCH THAT $n = 2k$.

$$3n = 3(2k) \quad \text{(SUBSTITUTION)}$$
$$= 6k$$
$$= 2(3k)$$

SINCE k IS AN INTEGER, $3k$ IS ALSO AN INTEGER. FROM THE DEFINITION OF EVEN, $2 \cdot$ INTEGER IS EVEN. THEREFORE $3n$ IS EVEN. SINCE THE CONTRAPOSITIVE WAS PROVEN TRUE, THEN THE ORIGINAL STATEMENT IS ALSO TRUE. ■

2. Prove using an indirect proof the following statement.

If $m \times n$ is even, then either m or n must be even.

$$\underbrace{\text{If } m \times n \text{ is even}}_{P} \longrightarrow \underbrace{\text{either m or n must be even}}_{Q}$$

$$\sim Q \longrightarrow \sim P$$

"IF m AND n ARE ODD, THEN $m \times n$ IS ODD."

TEST HYPOTHESIS: $m = 3$ $\quad n = 5$ $\quad m \cdot n = 3 \cdot 5 = 15$ ✓

$m = 11$ $\quad n = 3$ $\quad m \cdot n = 11 \cdot 3 = 33$ ✓

$m = 9$ $\quad n = 7$ $\quad m \cdot n = 9 \cdot 7 = 63$ ✓

PROOF: LET m BE ODD. FROM THE DEFINITION OF ODD, THERE EXISTS AN INTEGER L SUCH THAT $m = 2L + 1$. LET n ALSO BE ODD. FROM THE DEFINITION OF ODD, THERE EXISTS AN INTEGER K SUCH THAT $n = 2K + 1$.

$$m \cdot n = (2L + 1)(2K + 1) \quad \text{(SUBSTITUTION)}$$

$$= 4LK + 2K + 2L + 1$$

$$= 2(2LK + K + L) + 1$$

SINCE K & L ARE INTEGERS, $2LK + K + L$ IS ALSO AN INTEGER. FROM THE DEFINITION OF ODD, $2 \cdot$ INTEGER $+ 1$ IS ODD. $m \cdot n = 2 \cdot$ INTEGER $+ 1$ WHICH IS ODD. SINCE THE CONTRAPOSITIVE WAS TRUE, THEN THE ORIGINAL STATEMENT IS ALSO TRUE. ∎