1. Derivation

1.1. **Curvature.**

1.1.1. *Covariant Differentiation.* We start with a vector $\vec{V} = V^\alpha \vec{e_\alpha}$. In Cartesian coordinates, the derivative of $\vec{V}$ with respect to x^β is

$$\begin{aligned}\frac{\partial \vec{V}}{\partial x^\beta} &= \frac{\partial}{\partial x^\beta}(V^\alpha \vec{e_\alpha}) \\ &= \vec{e_\alpha}\frac{\partial V^\alpha}{\partial x^\beta}\end{aligned}$$

In an arbitrary coordinate system, the derivative of the basis vectors do not necessarily go to zero, and the derivative is

$$\frac{\partial \vec{V}}{\partial x^\beta} = \vec{e_\alpha}\frac{\partial V^\alpha}{\partial x^\beta} + V_\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta} \tag{1.1}$$

One can define

$$\vec{e}_\mu \Gamma^\mu{}_{\alpha\beta} \equiv \frac{\partial \vec{e}_\alpha}{\partial x^\beta}$$

in which Eqn. (1.1) simplifies to

$$\frac{\partial \vec{V}}{\partial x^\beta} = \vec{e_\alpha}\frac{\partial V^\alpha}{\partial x^\beta} + \vec{e}_\mu \Gamma^{\mu\alpha}{}_{\alpha\beta} \tag{1.2}$$

On the second term, we can exchange the indices μ and α, to get

$$\begin{aligned}\frac{\partial \vec{V}}{\partial x^\beta} &= \vec{e_\alpha}\frac{\partial V^\alpha}{\partial x^\beta} + \vec{e}_\alpha \Gamma^\alpha{}_{\mu\beta} V^\mu \\ &= \vec{e_\alpha}\left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\mu\beta} V^\mu\right)\end{aligned} \tag{1.3}$$

We represent a partial derivative of a vector as $V^\alpha{}_{,\beta}$. Similarly, the covariant derivative ($V^\alpha{}_{;\beta}$) is defined as the term in parentheses.

$$V^\alpha{}_{;\beta} = \frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\mu\beta} V^\mu \tag{1.4}$$

Our derivative is now expressed as

$$\frac{\partial \vec{V}}{\partial x^\beta} = V^\alpha{}_{;\beta}\vec{e_\alpha} \tag{1.5}$$

$\Gamma^\alpha{}_{\mu\beta}$ can be expressed in terms of the metric:

$$\Gamma^\gamma{}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}[g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}] \tag{1.6}$$

The reason one uses the covariant derivative is that in an arbitrary coordinate system, the basis vectors change with position. Normally, in Cartesian coordinates, the basis vectors are constant, and their derivatives zero. The term $\Gamma^\alpha{}_{\mu\beta}$ is known as the Christoffel symbols.

1.1.2. *Riemann Curvature.* In a gravitational field, a Lorentz frame cannot be globally defined. To show this, one can devise an experiment in which two light beams travel a congruent path in a gravitational field, such as the Earth's. One beam will travel downgradient (toward the Earth) and the other upgradient (away from Earth).The frequencies of the two waves can be measured experimentally, and are measured to be different. This shows that the periods of the waves ($T \equiv 1/\nu$) are different. The periods cannot be different in an inertial frame, and therefore the frame is not globally inertial.

A local Lorentz frame can be defined; if a frame is freely falling in a weak gravitational field, then locally, it is nearly Lorentz. As the gravitational field strengthens, the area that can be defined as locally Lorentz diminishes, because if it is extended too far in space, the edges will not be falling freely, and as it is extended in time, the field will strengthen and tighten the restrictions.

When a global Lorentz frame cannot be defined, we say that the spacetime is curved. (A global Lorentz frame, as in SR, would give a flat spacetime). We treat spacetime as a manifold: SR represents a flat manifold, while GR gives a curved manifold.) On the curved manifold, one can define local areas of flatness, where a frame would be nearly Lorentz. A metric $g_{\mu\nu}$ is defined on the manifold, generalized from $\eta_{\mu\nu} \equiv \delta_{\mu\nu} \neq g_{\mu\nu}$.

On a curved manifold, vectors do not parallel-transport themselves; if we move a vector from a point A to a point $A + \delta A$ so that the two vectors are parallel, and continue to a point B much farther away, then the vectors at A and B will not be parallel. We can extend this idea by constructing a loop on the manifold and parallel-transporting a vector around the loop; when it returns to the starting point, the new vector will not be parallel with the original vector.

For a vector V^α transported from point A to point B, the change is represented by

$$\begin{aligned} V^\alpha(B) &= V^\alpha(A) + \int \frac{\partial}{\partial x^\beta} V^\alpha dx^\beta \\ &= V^\alpha(A) - \int V^\mu {\Gamma^\alpha}_{\mu\beta} dx^\beta \end{aligned}$$

If we set up a "rectangular" loop with sides δa and δb then we can show that the change in the vector V^α is

$$\delta V^\alpha = \delta a \delta b [{\Gamma^\alpha}_{\mu\sigma,\lambda} - {\Gamma^\alpha}_{\mu\lambda,\sigma} + {\Gamma^\alpha}_{\nu\lambda}{\Gamma^\nu}_{\mu\sigma} - {\Gamma^\alpha}_{\nu\sigma}{\Gamma^\nu}_{\mu\lambda}] V^\mu$$

We will now define a tensor ${R^\alpha}_{\beta\mu\nu}$:

$$(1.7) \qquad {R^\alpha}_{\beta\mu\nu} \equiv {\Gamma^\alpha}_{\mu\sigma,\lambda} - {\Gamma^\alpha}_{\mu\lambda,\sigma} + {\Gamma^\alpha}_{\nu\lambda}{\Gamma^\nu}_{\mu\sigma} - {\Gamma^\alpha}_{\nu\sigma}{\Gamma^\nu}_{\mu\lambda}$$

$\mathbf{R} \equiv {R^\alpha}_{\beta\mu\nu}$ is known as the Riemann curvature tensor. When supplied with a vector V and the path taken, it returns the deviation δV. $\mathbf{R}$ characterizes the curvature on the manifold. In a locally Lorentz frame, $\mathbf{R}$ can be written in terms of the metric $g_{\mu\nu}$, using Eqn. (1.6):

$$(1.8) \qquad {R^\alpha}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma} [g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu}]$$

If we take a partial derivative (not covariant) of $\mathbf{R}$ and look at the components in a locally inertial frame (where Eqn. (1.8) is valid), we get

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2} [g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}]$$

The symmetry of the metric $g_{\mu\nu} = g_{\nu\mu}$ and the restriction to a Lorentz frame, where ${\Gamma^\mu}_{\alpha\beta} = 0$ gives us

$$\begin{aligned} R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} &= 0 \\ (1.9) \qquad R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} &= 0 \end{aligned}$$

Eqn. (1.9) is known as the Bianchi identities.

We also need to define $R_{\alpha\beta}$ and R, the Ricci tensor and the Ricci scalar. $R_{\alpha\beta}$ is a contraction of $\mathbf{R}$ on the first and third indices, where R ia a contraction on its indices:

$$\begin{aligned} R_{\alpha\beta} &\equiv {R^\mu}_{\alpha\mu\beta} &= g^{\lambda\mu} R_{\lambda\alpha\mu\beta} \\ R &\equiv {R^\alpha}_\alpha &= g^{\alpha\beta} R_{\alpha\beta} \end{aligned}$$

If we apply these contractions to the Bianchi identities, we get the contracted Bianchi identities:

$$\begin{aligned} g^{\alpha\mu}[R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}] &= 0 \\ (1.10) \qquad R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + {R^\mu}_{\beta\nu\lambda;\mu} &= 0 \end{aligned}$$

(1.11)

$$\begin{aligned} g^{\beta\nu}[R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + {R^\mu}_{\beta\nu\lambda;\mu}] &= 0 \\ (1.12) \qquad R_{;\lambda} - 2{R^\mu}_{\lambda;\mu} = 0 & \end{aligned}$$

By renaming indices and rearranging Eqn.(1.12), we can define a new tensor $G^{\alpha\beta}$in the following way:

$$
\begin{aligned}
R_{;\lambda} - 2R^{\mu}{}_{\lambda;\mu} &= 0 \\
-\frac{1}{2}\delta^{\mu}{}_{\lambda}R_{;\mu} + R^{\mu}{}_{\lambda;\mu} &= 0 \\
[R^{\mu}{}_{\lambda} - \frac{1}{2}\delta^{\mu}{}_{\lambda}R]_{;\mu} &= 0 \\
\mu &\rightarrow \beta \\
[R^{\beta}{}_{\lambda} - \frac{1}{2}\delta^{\beta}{}_{\lambda}R]_{;\beta} &= 0 \\
[g^{\alpha\lambda}R^{\beta}{}_{\lambda} - \frac{1}{2}g^{\alpha\lambda}\delta^{\beta}{}_{\lambda}R]_{;\beta} &= 0 \\
[R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R]_{;\beta} &= 0 \\
G^{\alpha\beta}{}_{;\beta} &= 0
\end{aligned}
$$

where $G^{\alpha\beta} \equiv [R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R] = G^{\beta\alpha}$

G is known as the Einstein tensor. It can be shown that $G^{\alpha\beta}$ can be simply related to the stress-energy tensor $T^{\alpha\beta}$.

We need to find the relation between the gravitational field and the curvature caused by it. In Newtonian gravity, the field ϕ is governed by the equation

$$\nabla^2\phi = 4\pi G\rho \tag{1.13}$$

relating the gravitational field to the mass density generating the gravitational field. (Henceforth, we shall take $G = c = 1$) In GR, the relationship should be analogous, and simplify to Newtonian gravity in the appropriate limits. We cannot use the mass density ρ and ignore the energy density, nor can we use the energy density T^{00}because T^{00} depends on the frame it is observed in. Therefore, we need an equation of the form

$$\mathbf{O}(\mathbf{g}) = k\mathbf{T} \tag{1.14}$$

with k constant, and **O** representing some differential operator $a_\eta(\frac{d}{dx^\alpha})^\eta, \eta \rightarrow 0...\infty$. Eqn.(1.13) is a second order differential equation; we should expect our equation to be similar, in that components of **O** should be linear combinations of $g_{\mu\nu},\ g_{\mu\nu,\alpha}$ and $g_{\mu\nu,\alpha\beta}$.

By enforcing the conservation law $T^{\alpha\beta}{}_{;\beta} = 0$, we require $O^{\alpha\beta}{}_{;\beta} = 0$. It can be shown that a combination of contractions of the Riemann curvature tensor and the metric meet all of these requirements; we can define $O^{\alpha\beta} = R^{\alpha\beta} + \mu g^{\alpha\beta}R + \Lambda g^{\alpha\beta}$. Since, in the local Lorentz frame in which $R^{\alpha\beta}$ and R were defined, $g^{\alpha\beta}{}_{;\beta} = 0$, we require that $(R^{\alpha\beta} + \mu g^{\alpha\beta}R)_{\beta} = 0$. The contracted Bianchi identities show that this is true for $\mu = \frac{1}{2}$, giving us

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta}\ . \tag{1.15}$$

These are Einstein's field equations. These equations relate the curvature to the stress-energy that causes the gravitational field. Generally, we use $\Lambda = 0$ and $k = 8\pi$.

1.2. **Linearized theory.**

1.2.1. *Weak gravitational fields.*

A weak gravitational field can be viewed as a perturbation of a flat spacetime, giving us a "nearly" flat spacetime. We can state this in equation form as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{1.16}$$

where $\eta_{\alpha\beta}$ is the metric of a flat spacetime, and $|h_{\alpha\beta}| \ll 1$ is the perturbation responsible for the gravitational field. By performing Lorentz transformations on $g_{\alpha\beta}$, we can show that $h_{\alpha\beta}$ transforms like a tensor in SR:

$$
\begin{aligned}
\Lambda^{\alpha}{}_{\alpha'}\Lambda^{\beta}{}_{\beta'}g_{\alpha\beta} &= \Lambda^{\alpha}{}_{\alpha'}\Lambda^{\beta}{}_{\beta'}\eta_{\alpha\beta} + \Lambda^{\alpha}{}_{\alpha'}\Lambda^{\beta}{}_{\beta'}h_{\alpha\beta} \\
g_{\alpha'\beta'} &= \eta_{\alpha'\beta'} + h_{\alpha'\beta'}
\end{aligned}
$$

where $h_{\alpha'\beta'} \equiv \Lambda^{\alpha}{}_{\alpha'}\Lambda^{\beta}{}_{\beta'}h_{\alpha\beta}$. This lets us treat $h_{\alpha\beta}$ and fields in our nearly flat spacetime, defined in terms of $h_{\alpha\beta}$, as fields in SR.

1.2.2. *Gauge transformations.* We shall define a transformation

$$x'_{\alpha} = x_{\alpha} + \xi_{\alpha}(x_{\beta})$$

Our transformation matrix is

$$\begin{aligned}\Lambda^{\alpha'}{}_{\beta} &= \delta^{\alpha}{}_{\beta} + \xi^{\alpha}{}_{,\beta} \\ \Lambda^{\alpha}{}_{\beta'} &= \delta^{\alpha}{}_{\beta} - \xi^{\alpha}{}_{,\beta}\,.\end{aligned}$$

Acting on $g_{\alpha\beta}$, we get

$$g_{\alpha'\beta'} = \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$

and we define

$$h'_{\alpha\beta'} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}\,.$$

With the requirement that$|\xi_{\alpha,\beta}|$ be small, our $h'_{\alpha\beta}$ remain a small perturbation, and we can still define our local Lorentz frame.

1.2.3. *Einstein's equations in terms of $h_{\alpha\beta}$.* Using Eqns. (1.8) and (1.16), we can show that

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}[h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu}] \tag{1.17}$$

If we define

$$\begin{aligned} h &\equiv h^{\alpha}{}_{\alpha} \\ \bar{h}^{\alpha\beta} &= h^{\alpha\beta} - \frac{1}{2}h\eta^{\alpha\beta}\end{aligned}$$

then we can state $G_{\alpha\beta}$ in terms of $\bar{h}_{\alpha\beta}$:

$$G_{\alpha\beta} = -\frac{1}{2}[\bar{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{,\mu} - \bar{h}_{\beta\mu,\alpha}{}^{,\mu}\,. \tag{1.18}$$

Our gauge freedom allows us to choose a gauge in which $\bar{h}^{\mu\nu}{}_{\nu} = 0$. We need to choose ξ^{α} so that $\Box\xi^{\alpha} = \bar{h}^{\alpha\beta}{}_{,\beta}$. This leaves us with

$$\begin{aligned} G_{\alpha\beta} &= -\frac{1}{2}\Box\bar{h}_{\alpha\beta} \\ \Box\bar{h}_{\mu\nu} &= -16\pi T_{\mu\nu} \qquad (1.19)\end{aligned}$$

($\Box$ represents the four-dimensional Laplacian, $\nabla^2 - \frac{\partial^2}{\partial t^2}$, also known as the d'Alambertian.)

1.3. **Gravitational Waves.**

1.3.1. *Solving the wave equation.* Einstein's equations representing a weak and fluctuating field, in a vacuum where $T^{\alpha\beta} = 0$ are

$$\Box\bar{h}^{\alpha\beta} = \bar{h}^{\alpha\beta}{}_{,\mu}{}^{,\mu} = \eta^{\mu\nu}\bar{h}^{\alpha\beta}{}_{,\mu\nu} = 0 \tag{1.20}$$

This is the equation for a gravitational wave generated by a distant source.

The solution to wave equations of the form

$$\Box f = 0$$

is known to be of the form $f = A\exp ikx$. This leads us to the solution

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta}\exp ik_{\mu}x^{\mu} \tag{1.21}$$

with $A^{\alpha\beta}$ (the amplitude of the wave) and k_{μ} (the wave number) constant.
From Eqn. (1.20) and Eqn. (1.21) we can derive

$$
\begin{aligned}
\bar{h}^{\alpha\beta}{}_{,\mu} &= k_\mu \bar{h}^{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
\eta^{\mu\nu}\bar{h}^{\alpha\beta}{}_{,\mu\nu} &= k_\mu k_\nu \bar{h}^{\alpha\beta} = 0 \\
k^\mu k_\mu \bar{h}^{\alpha\beta} &= 0 \\
k^\mu k_\mu &= 0 \qquad (1.22)
\end{aligned}
$$

which shows that k is a null vector: $\omega^2 = {k_0}^2 = k_a k^a$ where ω represents the frequency of the wave.

In stating the wave equation, we assumed $\bar{h}^{\alpha\beta}{}_{,\beta} = 0$, from which we derive

$$
\begin{aligned}
\bar{h}^{\alpha\beta} &= A^{\alpha\beta} \exp ik_\mu x^\mu \\
\bar{h}^{\alpha\beta}{}_{,\beta} &= iA^{\alpha\beta} k_\beta \exp ik_\mu x^\mu \\
&= ik_\beta \bar{h}^{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
iA^{\alpha\beta} k_\beta \exp ik_\mu x^\mu &= 0 \\
A^{\alpha\beta} k_\beta &= 0 \qquad (1.23)
\end{aligned}
$$

From our assumption that $\bar{h}^{\alpha\beta}{}_{,\beta} = 0$, we now have the restriction that $A^{\alpha\beta}$ and $\overrightarrow{k}$ are perpendicular.

1.3.2. *Transverse-Traceless Gauge.* We gain further restrictions on $A^{\alpha\beta}$ from our gauge condition. We need to choose an ξ_α as a gauge transformation, such that $\Box \xi_\alpha = 0$, so that the resulting $h'^{\alpha\beta}$ still satisfies our wave equation:

$$
\begin{aligned}
\Box h'_{\alpha\beta} &= \Box(h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}) \\
&= \Box h_{\alpha\beta} - \Box\xi_{\alpha,\beta} - \Box\xi_{\beta,\alpha} \\
&= \Box h_{\alpha\beta} - (\Box\xi_\alpha)_{,\beta} - (\Box\xi_\beta)_{,\alpha} \\
&= \Box h_{\alpha\beta}
\end{aligned}
$$

The solution for $\Box\xi_\alpha = 0$ is of the same form as Eqn. (1.21): $\xi_\alpha = B_\alpha \exp ik_\mu x^\mu$, with B_α constant. Our $h_{\alpha\beta}$ is changed to give us

$$
\begin{aligned}
h'_{\alpha\beta} &= h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \\
\bar{h}'_{\alpha\beta} &= \bar{h}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi^\mu{}_{,\mu} \\
A'^{\alpha\beta} \exp ik_\beta x^\beta &= A^{\alpha\beta} \exp ik_\beta x^\beta - ik_\beta B_\alpha \exp ik_\beta x^\beta - ik_\alpha B_\beta \exp ik_\beta x^\beta + ik_\mu B^\mu \exp ik_\beta x^\beta
\end{aligned}
$$

$$
A'^{\alpha\beta} = A^{\alpha\beta} - ik_\beta B_\alpha - ik_\alpha B_\beta + ik_\mu B^\mu
$$

We can further restrict $A'_{\alpha\beta}$ by judicious choice of B_α to require

$$
\begin{aligned}
A_{\alpha\beta} U^\beta &= 0 \quad (Transverse) \qquad (1.24) \\
A^\alpha{}_\alpha &= 0 \quad (Traceless) \qquad (1.25)
\end{aligned}
$$

for a 4-velocity $\overrightarrow{U}$.

In a Lorentz frame, where $U^\alpha = \delta^\alpha{}_0$, and the orientation is such that our wave travels along the z axis $\overrightarrow{k} = (\omega, 0, 0, \omega)$ (nullity of $\overrightarrow{k}$ requires $k_0 = k_z$), Eqn. (1.23) and Eqn. (1.24) impose the requirement $A_{\alpha z} = A_{\alpha 0} = 0$, Eqn. (1.25) requires that $A_{yy} = -A_{xx}$, and symmetry of $g_{\alpha\beta}$ (and therefore $h_{\alpha\beta}$ and $A_{\alpha\beta}$) requires $A_{xy} = A_{yx}$. Out of 16 terms in $\mathbf{A}$, only A_{xx} and A_{xy} are free; they represent the components of the amplitude in two polarizations: A_{xx} and $A_{yy} = -A_{xx}$ (known as A_+) represent polarization along the x and y axes, while $A_{xy} = A_{yx}$ (known as $A_\times$)represent polarization at an angle $(\pi/4)$ to the axes.

1.4. **Generation of Gravitational Waves.** To solve for the generation of gravitational waves, we intend to solve Eqn. (1.19):

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{1.26}$$

1.4.1. *Simplified case.* We shall assume that the source of the gravitational wave is periodic. Since most detectable sources will be roughly periodic, and any time dependence can be made by a suitable sum of sinusoidal sources, we can make this assumption without loss of generality. Therefore, we assume

$$T_{\mu\nu} = S_{\mu\nu}(x^i)exp(i\Omega t)$$

We shall also assume that the radius in which $S_{\mu\nu} \neq 0$ is small compared to a wavelength of the gravitational wave $\varepsilon \ll (2\pi/\Omega)$. This limits the velocity of the source to be much less than 1. This assumption (known as the slow-motion assumption) fits most sources of gravitational waves.

We will look for the solution

$$\bar{h}_{\mu\nu} = B_{\mu\nu}(x^i)exp(i\Omega t)$$

Plugging into Eqn. (1.26),

$$\begin{aligned}
(\nabla^2 - \frac{\partial^2}{\partial t^2})B_{\mu\nu}exp(i\Omega t) &= -16\pi S_{\mu\nu}exp(i\Omega t) \\
(\nabla^2 - i^2\Omega^2)B_{\mu\nu}exp(i\Omega t) &= -16\pi S_{\mu\nu}exp(i\Omega t) \\
(\nabla^2 + \Omega^2)B_{\mu\nu} &= -16\pi S_{\mu\nu} \qquad (1.27)
\end{aligned}$$

A solution of this equation is known to be of the form

$$B_{\mu\nu} = \frac{A_{\mu\nu}}{r}exp(i\Omega r) + \frac{Z_{\mu\nu}}{r}exp(-i\Omega r) \tag{1.28}$$

with constant $A_{\mu\nu}$ and $Z_{\mu\nu}$, and r is the standard spherical radial coordinate, with the center chosen inside the source. The term $exp(-i\Omega r)$ represents an wave coming in from $r \to +\infty$, while the term $exp(+i\Omega r)$ describes a wave leaving the source from $r = 0$. Since we want the waves generated from our source, we must choose $Z_{\mu\nu} = 0$

Going back to Eqn. (1.27), we will make our slow-motion assumption and integrate over the volume of a sphere with radius $\varepsilon \ll (2\pi/\Omega)$

$$\int \nabla^2 B_{\mu\nu}d^3x + \int \Omega^2 B_{\mu\nu}d^3x = -16\pi \int S_{\mu\nu}d^3x \tag{1.29}$$

Solving for each term,

$$\int \Omega^2 B_{\mu\nu}d^3x \leq \Omega^2|B_{\mu\nu}|_{max}\frac{4\pi\varepsilon^3}{3} \tag{1.30}$$

$$\begin{aligned}
\int \nabla^2 B_{\mu\nu}d^3x &= \oint \mathbf{n}\cdot\nabla B_{\mu\nu}dS \qquad (1.31) \\
&= 4\pi\varepsilon^2\frac{d}{dr}B_{\mu\nu}\bigg|_{r=\varepsilon} \\
&= -4\pi A_{\mu\nu}
\end{aligned}$$

$$J_{\mu\nu} \equiv \int S_{\mu\nu}d^3x \tag{1.32}$$

The second term that generates Eqn. (1.30) is negligible when compared to the other terms, and will be dropped, giving us

$$\begin{aligned}
A_{\mu\nu} &= 4J_{\mu\nu} \\
\bar{h}_{\mu\nu} &= 4J_{\mu\nu}\frac{exp(i\Omega[r-t])}{r} \qquad (1.33)
\end{aligned}$$

To rid ourselves of $J_{\mu\nu}$,

$$\begin{aligned}
J_{\mu\nu} exp(-i\Omega t) &= \int T_{\mu\nu} d^3x \\
\frac{d}{dt} J_{\mu\nu} exp(-i\Omega t) &= \frac{d}{dt} \int T_{\mu\nu} d^3x \\
-i\Omega J_{\mu 0} exp(-i\Omega t) &= \int T_{\mu 0}{}^{,0} d^3x \qquad (1.34)
\end{aligned}$$

Because $T_{\mu\nu}$ is divergence-free ($T^{\mu\nu}{}_{,\nu} = 0$) we can replace $T_{\mu 0}{}^{,0}$ with $-T_{\mu k}{}^{,}k$ to get

$$\begin{aligned}
i\Omega J_{\mu 0} exp(-i\Omega t) &= \int T_{\mu k}{}^{,k} d^3x \\
&= \oint T_{\mu k} n^k dS = 0
\end{aligned}$$

$T_{\mu k} n_k dS = 0$ is true because we apply Gauss' Law so that we integrate over a volume completely containing the source, where $T_{\mu\nu} = 0$ on the Gaussian surface. If $\Omega \neq 0$ (otherwise, there would be no wave generated) then $J_{\mu 0} = \bar{h}_{\mu 0} = 0$.

To solve for J_{ij}, we need to solve for T_{ij}:

$$\begin{aligned}
J_{ij} exp(-i\Omega t) &= \int T_{ij} d^3x \\
&= \frac{1}{2} \frac{d}{dt}^2 \int T_{00} x_i x_j d^3x \quad (from\ tensor\ virial\ theorem) \\
&\approx \frac{1}{2} \frac{d}{dt}^2 \int \rho x_i x_j d^3x \\
&= \frac{1}{2} \ddot{I}_{ij}(t) \qquad (1.35) \\
\bar{h}_{ij} &= 2 \ddot{I}_{ij} \frac{exp(i\Omega r)}{r} \qquad (1.36)
\end{aligned}$$

where $I_{ij} \equiv \int T_{00} x_i x_j d^3x$ is the quadrupole moment of inertia tensor, and $\ddot{I}_{ij}$ is the second time derivative of I_{ij}. Making the replacement $I_{ij} \equiv D_{ij} exp(-i\Omega t)$ we get

$$\begin{aligned}
\bar{h}_{ij} &= 4 \left[\frac{1}{2} \left(\frac{d}{dt} \right)^2 D_{ij} exp(-i\Omega t) \right] \frac{exp(i\Omega r)}{r} \\
&= -2\Omega^2 D_{ij} \frac{exp(i\Omega [r-t])}{r} \qquad (1.37)
\end{aligned}$$

We can now apply our gauge conditions to $\bar{h}_{\mu\nu}$ to simplify everything to $\bar{h}_{xx}$ and $\bar{h}_{xy}$.for a wave to travel in the z-direction at the point of measurement.

$$\begin{aligned}
\bar{h}^{TT}_{xx} &= 2\ddot{I}_{xx}\frac{exp(i\Omega r)}{r} \\
&= 2\left[\left(\frac{d}{dt}\right)^2 D_{xx}exp(-i\Omega t)\right]\frac{exp(i\Omega r)}{r} \\
&= -2\Omega^2 D_{xx}exp(-i\Omega t)\frac{exp(i\Omega r)}{r} \\
&= -2\Omega^2 I_{xx}\frac{exp(i\Omega r)}{r} \qquad (1.38)
\end{aligned}$$

$$\begin{aligned}
\bar{h}^{TT}_{xy} &= 2\ddot{I}_{xy}\frac{exp(i\Omega r)}{r} \\
&= 2\left[\left(\frac{d}{dt}\right)^2 D_{xy}exp(-i\Omega t)\right]\frac{exp(i\Omega r)}{r} \\
&= -2\Omega^2 D_{xy}exp(-i\Omega t)\frac{exp(i\Omega r)}{r} \\
&= -2\Omega^2 I_{xy}\frac{exp(i\Omega r)}{r} \qquad (1.39)
\end{aligned}$$

We can write Eqns.(1.38) and (1.39) in terms of the reduced moment if inertia tensor: $\mathcal{I}_{ij} \equiv I_{ij} - \frac{1}{3}\delta_{ij}I^k{}_k$. With this form (which is trace free), we can show that

$$\begin{aligned}
\mathcal{I}_{xy} &= I_{xy} - \frac{1}{3}\delta_{xy}I^k{}_k \\
&= I_{xy}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{xx} - \mathcal{I}_{y}y &= [I_{xx} - \frac{1}{3}\delta_{xx}I^k{}_k] - [I_{yy} - \frac{1}{3}\delta_{yy}I^k{}_k] \\
&= I_{xx} - I_{yy}
\end{aligned}$$

$$\begin{aligned}
\bar{h}^{TT}_{xx} &= -2\Omega^2 I_{xx}\frac{exp(i\Omega r)}{r} \\
\bar{h}^{TT}_{yy} &= -2\Omega^2 I_{yy}\frac{exp(i\Omega r)}{r} = -\bar{h}^{TT}_{xx}
\end{aligned}$$

$$\begin{aligned}
I_{xx} &= -I_{yy} \\
2I_{xx} &= I_{xx} - I_{yy}
\end{aligned}$$

$$\bar{h}^{TT}_{xx} = -\Omega^2[I_{xx} - I_{yy}]\frac{exp(i\Omega r)}{r}$$

$$\bar{h}^{TT}_{xx} = -\Omega^2[\mathcal{I}_{xx} - \mathcal{I}_{yy}]\frac{exp(i\Omega r)}{r} \qquad (1.40)$$

$$\bar{h}^{TT}_{xy} = -2\Omega^2\mathcal{I}_{xy}\frac{exp(i\Omega r)}{r} \qquad (1.41)$$

We now have, for a gravitational wave (in the z direction), $\bar{h}_{xx}$ and $\bar{h}_{xy}$ (representing the components of the wave in the + and $\times$ polarizations in terms of the moment of inertia of the source $\mathcal{I}_{xx}$ and $\mathcal{I}_{xy}$.

1.4.2. *Overview of exact solution.* Wave equations like Eqn. (1.20) are known in general to have the outgoing-wave solution

$$\bar{h}_{\mu\nu}(t, x^i) = 4 \int \frac{T_{\mu\nu}(t - R, y^i)}{R} d^3y \tag{1.42}$$

for arbitrary $T_{\mu\nu}$ where $R = |x^i - y^i|$, x^i is the field point, y^i is the source point, the origin is centered in the source, and $t - R$ represents the retarded time. (The value of $h_{\mu\nu}$ at the field point at time t depends on events at the source point at time $t - R$, because the information must travel over the distance at speed c.) Measuring at a field point far away from the source ($|x^i| = r$ is much greater than $|y^i|$) we can assume that $R \approx r$ to get

$$\bar{h}_{\mu\nu}(t, x^i) = \frac{4}{r} \int T_{\mu\nu}(t - R, y^i) d^3y \tag{1.43}$$

Utilizing the conservation law $T^{\mu\nu}{}_{,\nu} = 0$, and making our gauge choice as previously, we can reduce $\bar{h}_{\mu\nu}$ to

$$\bar{h}^{TT}_{xx} = \frac{1}{r}[\ddot{\mathcal{I}}_{xx}(t - r) - \ddot{\mathcal{I}}_{yy}(t - r)] \tag{1.44}$$

$$\bar{h}^{TT}_{xy} = \frac{2}{r}[\ddot{\mathcal{I}}_{xy}(t - r)] \tag{1.45}$$

1.5. **Energy in gravitational waves.** We are going to contrive a setup for measuring the energy contained in gravitational waves. A gravitational wave of the form $\bar{h}^{TT}_{xx} = A\,cos(\Omega[z - t])$ is incident on a plane of resonant detectors of density σ. Each resonant detector is made up of two particles of mass m at positions x_1 and x_2, on a spring with unstretched length l_0, spring constant k and dampening constant ν. The gravitational wave will cause each detector to oscillate, so that the proper length l at any instant is given by

$$l(t) = \int_{x_1(t)}^{x_2(t)} \sqrt{1 + \bar{h}^{TT}_{xx}(t)}\, dx\,.$$

The differential equations governing the positions of the particles are

$$\begin{aligned} m\ddot{x}_1 &= -k(l_0 - l) - \nu\frac{d}{dt}[l_0 - l] \\ m\ddot{x}_2 &= -k(l - l_0) - \nu\frac{d}{dt}[l - l_0] \end{aligned}$$

By defining $\xi \equiv l - l_0$, $\gamma \equiv \frac{\nu}{m}$, and $\omega_0^2 \equiv \frac{2k}{m}$, we can simplify to the standard form for a damped driven oscillator

$$\ddot{\xi} + 2\gamma\dot{\xi} + \omega_0^2\xi = \frac{1}{2}l_0\left(\frac{d}{dt}\right)^2 \bar{h}^{TT}_{xx} \tag{1.46}$$

The steady solution to Eqn.(1.46) is known to be

$$\xi = Rcos[\Omega t + \phi] \tag{1.47}$$

$$\begin{aligned} R &\equiv \frac{\frac{1}{2}l_0\Omega^2 A}{[(\omega_0^2 - \Omega^2)^2 + 4\Omega^2\gamma^2]^{\frac{1}{2}}} \\ tan\phi &= \frac{2\gamma\Omega}{(\omega_0 - \Omega)^2} \end{aligned}$$

for our choice of h^{TT}_{xx}.

The losses from dampening are cancelled by the energy from the wave driving the oscillation. Therefore, the power imparted to the oscillators by the wave is

$$\frac{dE}{dt} = \nu(\frac{d\xi}{dt})^2 = m\gamma(\frac{d\xi}{dt})^2\,.$$

Averaging over a period of oscillation gives us

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2}m\gamma R^2\Omega^2\,.$$

The total energy flux across the array of oscillators (σ per unit area) is

$$\delta F = -\sigma \left\langle \frac{dE}{dt} \right\rangle = -\frac{1}{2} m\gamma\sigma R^2\Omega^2 \,. \tag{1.48}$$

The oscillators themselves radiate gravitational waves; each has a moment of inertia $I_{xx} = ml_0\xi = ml_0Rcos[\Omega t + \phi]$. It can be shown that our array of oscillators emit a gravitational wave

$$\delta\bar{h}^{TT}_{xx} = 2\pi\sigma m\Omega l_0 Rsin[\Omega[z-t]-\phi] \tag{1.49}$$

and the net wave beyond the detectors is

$$\bar{h}^{net}_{xx} = (A - 2\pi\sigma m\Omega l_0 Rsin(\phi))cos[\Omega[z-t]-\psi] \tag{1.50}$$

where ψ is a phase factor given by $tan\psi = \frac{2\pi\sigma m\Omega l_0 R}{A}cos(\phi)$

This gives us a change in the amplitude of the wave responsible for the flux change at or oscillators

$$\delta A = -2\pi\sigma m\Omega l_0 Rsin(\phi) \tag{1.51}$$

We can solve for the ratio of the change in flux to the change in amplitude:

$$\begin{aligned}
\frac{\delta F}{\delta A} &= \frac{-\frac{1}{2}m\gamma\sigma R^2\Omega^2}{-2\pi\sigma m\Omega l_0 Rsin(\phi)} \\
&= \frac{1}{4\pi}\frac{\gamma R\Omega}{l_0 sin\phi} \\
&= \frac{1}{4\pi}\frac{\frac{1}{2}l_0\Omega^2 A}{([\omega_0^2-\Omega^2]^2-4\gamma^2\Omega^2)^{\frac{1}{2}}}\frac{\gamma\Omega}{l_0}\frac{1}{sin\phi} \\
&= \frac{\Omega^2 A}{8\pi}\frac{1}{cos\phi}\frac{\omega_0^2-\Omega^2}{2\gamma\Omega}\frac{1}{([\omega_0^2-\Omega^2]^2-4\gamma^2\Omega^2)^{\frac{1}{2}}} \\
&= \frac{\Omega^2 A}{16\pi}\frac{1}{cos\phi}\frac{\omega_0^2-\Omega^2}{([\omega_0^2-\Omega^2]^2-4\gamma^2\Omega^2)^{\frac{1}{2}}} \\
&= \frac{\Omega^2 A}{16\pi}\frac{1}{cos\phi}cos\phi \\
\frac{\delta F}{\delta A} &= \frac{1}{16\pi}\Omega^2 A
\end{aligned} \tag{1.52}$$

If we integrate Eqn. (1.52), we get the total flux

$$F = \frac{1}{32\pi}\Omega^2 A^2 \tag{1.53}$$

We substitute $\bar{h}^{TT}_{\mu\nu}$ for A:

$$\begin{aligned}
\frac{1}{2}A^2 &= \langle \bar{h}^{TT2}_{xx} \rangle \\
\frac{1}{2}A^2 &= \langle \bar{h}^{TT2}_{yy} \rangle = \langle (-\bar{h}^{TT}_{xx})^2 \rangle \\
A^2 &= \langle h^{TT2}_{\mu\nu} \rangle = \langle \bar{h}^{TT}_{\mu\nu}\bar{h}^{\mu\nu}_{TT} \rangle \\
F &= \frac{1}{32\pi}\langle \bar{h}^{TT}_{\mu\nu}\bar{h}^{\mu\nu}_{TT} \rangle
\end{aligned} \tag{1.54}$$

1.5.1. *Energy loss from a source due to gravitational radiation.* The luminosity source is equal to the flux from the source integrated over a surface enveloping the source:

$$L = \int F dS \,.$$

We can show from either Eqns. (1.40) and(1.41), or (1.44) and (1.45) that

$$\bar{h}^{TT}_{jk} = -\frac{2}{r}\Omega^2\mathcal{I}_{jk} = \frac{2}{r}\ddot{\mathcal{I}}_{jk} \,.$$

and the flux from the source generating waves of this form is

$$F = -\frac{\Omega^6}{32\pi r^2}\left\langle \left[\frac{2}{r}\ddot{\mathcal{I}}_{jk}\right]^2 \right\rangle .$$

For our source, a measurement along the z axis will give

$$\begin{aligned}
F &= \frac{\Omega^2}{32\pi}\left\langle (\bar{h}^{TT}_{xx})^2 + (-\bar{h}^{TT}_{xx})^2 + 2(\bar{h}^{TT}_{xy})^2 \right\rangle \\
&= \frac{\Omega^6}{32\pi r^2}\left\langle 2(4\mathcal{I}^2_{xx} + 4\mathcal{I}^2_{xy}) \right\rangle \\
&= \frac{\Omega^6}{32\pi r^2}\left\langle 2([\mathcal{I}_{xx} - \mathcal{I}_{yy}]^2 + 4\mathcal{I}^2_{xy}) \right\rangle \\
&= \frac{\Omega^6}{32\pi r^2}\left\langle (2[\mathcal{I}_{xx} - \mathcal{I}_{yy}]^2 + 8\mathcal{I}^2_{xy}) \right\rangle \qquad (1.55)
\end{aligned}$$

This can be rewritten, using ${\mathcal{I}^k}_k = 0$:

$$F = \frac{\Omega^6}{16\pi r^2}\left\langle 2\mathcal{I}_{ij}\mathcal{I}^{ij} - 4\mathcal{I}_{zj}{\mathcal{I}_z}^j + \mathcal{I}^2_{zz} \right\rangle$$

and generalized for any point a distance r form the source

$$F = \frac{\Omega^6}{16\pi r^2}\left\langle 2\mathcal{I}_{ij}\mathcal{I}^{ij} - 4n^j n^k \mathcal{I}_{ji}{\mathcal{I}_k}^i + n^i n^j n^k n^l \mathcal{I}_{ij}\rangle_{kl} \right\rangle$$

Integrating on the surface of a sphere of radius r, we get

$$\begin{aligned}
L &= \frac{1}{4}\Omega^6 16\pi r^2 \left\langle 2\mathcal{I}_{ij}\mathcal{I}^{ij} - \frac{4}{3}\mathcal{I}_{ij}{\mathcal{I}_k}^i + \frac{1}{1}5({\mathcal{I}^i}_i{\mathcal{I}^k}_k + \mathcal{I}^{ij}\mathcal{I}_{ij} \right\rangle \\
L &= \frac{1}{5}\Omega^6 \left\langle \mathcal{I}_{ij}\mathcal{I}^{ij} \right\rangle \qquad (1.56)
\end{aligned}$$

We can generalize further by exchanging factors of $i\Omega$ for time derivatives:

$$L = \frac{1}{5}\left\langle \frac{d^3}{dt^3}\left(\mathcal{I}_{ij}\right)^2 \right\rangle \qquad (1.57)$$

This is the general formula for the luminosity of a gravitational wave source. (This is good only in the approximations that we have previously made.)