Some Exercises on Limits and Partial Limits

This solutions document is an ongoing project. If you would like to contact the author with any questions, comments or suggestions about the solutions that have been provided here, please write to

lewins@mindspring.com

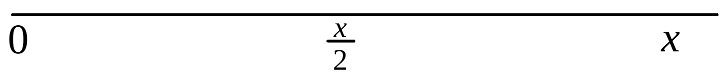
You may also send questions or comments in the form of attachments to your e-mail letters. If the document contains pictures, remember to save it as a* RAP FILE *before you send it. Please use a utility such as* PKZIP *to zip all files before you send them.

Note that hyperlinks that appear below in the* statement *of any exercise are not designed to work in this document. To operate those that appear in the statement of an exercise, go back to the exercises in the text. On the other hand, hyperlinks that appear in the* solution *of an exercise should work properly. Don't forget that you may need to hold down your control key when clicking on a hyperlink.

1. Given that
$$x_n = 3 + \frac{1}{n}$$
for each positive integer n, prove that 3 is a limit of (x_n).

2. Given that
$$x_n = 3 + \frac{2}{n}$$
for each positive integer n, prove that 3 is a limit of (x_n).

3. Given that $x_n = 1/n$ for each positive integer n and that $x \neq 0$, prove that x is not a partial limit of (x_n).
Solution: ***In the event that*** $x < 0$, ***the interval*** $(-\infty, 0)$ ***is a neighborhood of*** x ***and it is clear that*** (x_n) ***fails to be frequently in this neighborhood. Therefore no negative number can be a partial limit of*** (x_n). ***Suppose now that*** $x > 0$. ***The interval*** $(x/2, \infty)$ ***is a neighborhood of*** x



and the condition $x_n \in (x/2, \infty)$ ***must fail to hold whenever*** $n > 2/x$. ***Therefore the sequence*** (x_n) ***cannot be frequently in the interval*** $(x/2, \infty)$ ***and the number*** x ***cannot be a partial limit of*** (x_n).

4. Given that
$$x_n = \begin{cases} (-1)^n n^3 & \text{if } n \text{ is a multiple of 3} \\ 0 & \text{if } n \text{ is one more than a multiple of 3} \\ 4 & \text{if } n \text{ is two more than a multiple of 3} \end{cases},$$
Prove that the partial limits of (x_n) are $-\infty$, ∞, 0 and 4.

5. Give an example of a sequence of real numbers whose set of partial limits is the set $\{1\} \cup [4, 5]$.
Hint: ***For each positive integer*** n, ***if*** n ***can be written in the form***
$$n = 2^m 3^k$$
for some positive integers m ***and*** k ***and if***
$$4 \leq \frac{m}{k} \leq 5$$
then we define
$$x_n = \frac{m}{k}.$$
In all other cases we define $x_n = 1$. ***Observe that the range of the sequence*** (x_n) ***is the set***

$$\{1\} \cup (\mathbf{Q} \cap [4,5])$$

and then show that the set of partial limits of (x_n) ***is*** $\{1\} \cup [4,5]$.

6. Given that

$$x_n = \frac{3+2n}{5+n}$$

for every positive integer n, prove that $x_n \to 2$ as $n \to \infty$.

7. Given that

$$x_n = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is even} \\ \frac{1}{n^2+1} & \text{if } n \text{ is odd} \end{cases}$$

prove that $x_n \to 0$ as $n \to \infty$.

8. Suppose that (x_n) is a sequence of real numbers and that $x \in \mathbf{R}$. Prove that the following conditions are equivalent:

   a. $x_n \to x$ as $n \to \infty$.

   b. For every number $\varepsilon > 0$ the sequence (x_n) is eventually in the interval $(x - 5\varepsilon, x + 5\varepsilon)$.

9. Prove that

$$\frac{n^2+3n+1}{2n^2+n+4} \to \frac{1}{2}$$

as $n \to \infty$.

10. For each positive integer n, if n can be written in the form

$$n = 2^m 3^k$$

where m and k are postive integers and $m \le k$ then we define

$$x_n = \frac{m}{k}.$$

Otherwise we define $x_n = 0$. Prove that the set of partial limits of the sequence (x_n) is $[0,1]$.