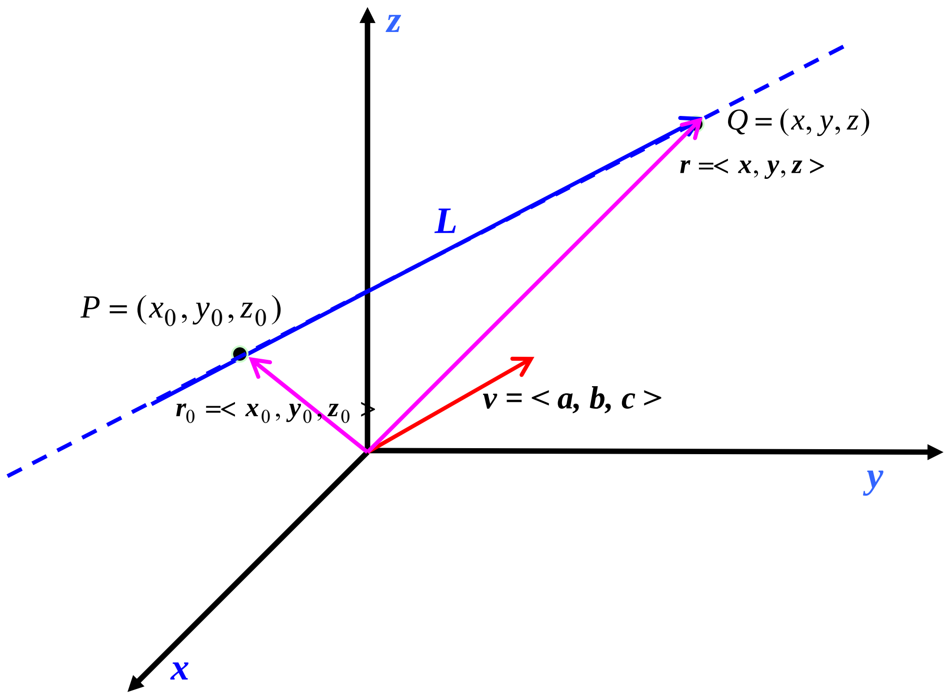
Section 9.5: Equations of Lines and Planes

Practice HW from Stewart Textbook (not to hand in)
p. 673 # 3-15 odd, 21-37 odd, 41, 47

Lines in 3D Space

Consider the line L through the point $P = (x_0, y_0, z_0)$ that is parallel to the vector $\mathbf{v} = < a, b, c >$



The line L consists of all points $Q = (x, y, z)$ for which the vector $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$.

Now,

$$\overrightarrow{PQ} = < x - x_0, y - y_0, z - z_0 >$$

Since $\overrightarrow{PQ}$ is parallel to $\mathbf{v} = < a, b, c >$,

$$\overrightarrow{PQ} = t\,\mathbf{v}$$

where t is a scalar. Thus

$$< \boldsymbol{x} - \boldsymbol{x}_0, \boldsymbol{y} - \boldsymbol{y}_0, \boldsymbol{z} - \boldsymbol{z}_0 >= \overrightarrow{\boldsymbol{PQ}} = t\, \boldsymbol{v} = < t\,a, t\,b, t\,c >$$

Rewriting this equation gives

$$< x, y, z > - < x_0, y_0, z_0 >= t < a, b, c >$$

Solving for the vector $< x, y, z >$ gives

$$< x, y, z >=< x_0, y_0, z_0 > + \; t < a, b, c >$$

Setting $\boldsymbol{r} = < x, y, z >$, $\boldsymbol{r}_0 = < x_0, y_0, z_0 >$, and $\boldsymbol{v} = < a, b, c >$, we get the following *vector equation* of a line.

Vector Equation of a Line in 3D Space

The vector equation of a line in 3D space is given by the equation

$$\boldsymbol{r} = \boldsymbol{r}_0 + \; t\, \boldsymbol{v}$$

where $\boldsymbol{r}_0 = < x_0, y_0, z_0 >$ is a vector whose components are made of the point (x_0, y_0, z_0) on the line L and $\boldsymbol{v} = < a, b, c >$ are components of a vector that is parallel to the line L.

If we take the vector equation

$$< x, y, z >=< x_0, y_0, z_0 > + \; t < a, b, c >$$

and rewrite the right hand side of this equation as one vector, we obtain

$$< x, y, z >=< x_0 + ta, y_0 + tb, z_0 + tc >$$

Equating components of this vector gives the *parametric equations of a line.*

Parametric Equations of a Line in 3D Space

The parametric equations of a line *L* in 3D space are given by

$$x = x_0 + ta, \quad y = y_0 + tb, \; z = z_0 + tc,$$

where (x_0, y_0, z_0) is a point passing through the line and $\mathbf{v} = < a, b, c >$ is a vector that the line is parallel to. The vector $\mathbf{v} = < a, b, c >$ is called the *direction vector* for the line *L* and its components *a, b,* and *c* are called the *direction numbers*.

Assuming $a \neq 0, b \neq 0, c \neq 0$, if we take each parametric equation and solve for the variable *t*, we obtain the equations

$$t = \frac{x - x_0}{a}, \qquad t = \frac{y - y_0}{b}, \qquad t = \frac{z - z_0}{c}$$

Equating each of these equations gives the *symmetric equations of a line*.

Symmetric Equations of a Line in 3D Space

The symmetric equations of a line *L* in 3D space are given by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where (x_0, y_0, z_0) is a point passing through the line and $\mathbf{v} = < a, b, c >$ is a vector that the line is parallel to. The vector $\mathbf{v} = < a, b, c >$ is called the *direction vector* for the line *L* and its components *a, b,* and *c* are called the *direction numbers*.

Note!! To write the equation of a line in 3D space, we need a point on the line and a parallel vector to the line.

Example 1: Find the vector, parametric, and symmetric equations for the line through the point (1, 0, -3) and parallel to the vector 2 ***i*** - 4 ***j*** + 5 ***k***.

Example 2: Find the parametric and symmetric equations of the line through the points (1, 2, 0) and (-5, 4, 2)

Solution: To find the equation of a line in 3D space, we must have at least one point on the line and a parallel vector. We already have two points one line so we have at least one. To find a parallel vector, we can simplify just use the vector that passes between the two given points, which will also be on this line. That is, if we assign the point P = (1, 2, 0) and Q = (-5, 4, 2), then the parallel vector v is given by

$$\boldsymbol{v} = \overrightarrow{\boldsymbol{PQ}} = < -5-1, 4-2, 2-0 > = < -6, 2, 2 >$$

Recall that the parametric equations of a line are given by

$$x = x_0 + ta, \quad y = y_0 + tb, \; z = z_0 + tc\,.$$

We can use either point P or Q as our point on the line (x_0, y_0, z_0). We choose the point P and assign $(x_0, y_0, z_0) = (1,2,0)$. The terms a, b, and c are the components of our parallel vector given by $\boldsymbol{v}$ = < -6, 2, 2 > found above. Hence a = -6, b = 2, and c = 2. Thus, the parametric equation of our line is given by

$$x = 1 + t\,(-6), \quad y = 2 + t\,(2), \; z = 0 + t\,(2)$$

or

$$\boxed{x = 1 - 6t, \quad y = 2 + 2t, \; z = 2t}$$

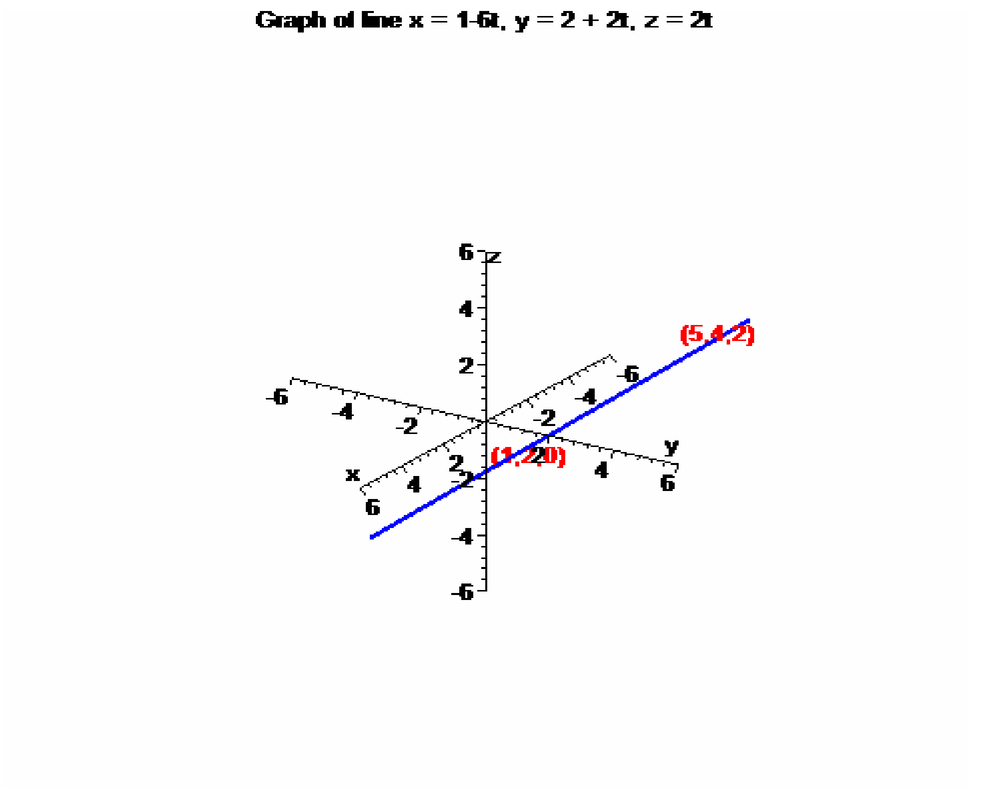
To find the symmetric equations, we solve each parametric equation for t. This gives

$$t = \frac{x-1}{-6}, \qquad t = \frac{y-2}{2}, \qquad t = \frac{z}{2}$$

Setting these equations equal gives the symmetric equations.

$$\boxed{\frac{x-1}{-6} = \frac{y-2}{2} = \frac{z}{2}}$$

The graph on the following page illustrates the line we have found



It is important to note that the equations of lines in 3D space are not unique. In Example 2, for instance, had we used the point Q = (-5, 4, 2) to represent the equation of the line with the parallel vector $\boldsymbol{v}$ = < -6, 2, 2 >, the parametric equations becomes

$$x = -5 - 6t, \quad y = 4 + 2t, \; z = 2 + 2t$$

Example 3: Find the parametric and symmetric equations of the line passing through the point (-3, 5, 4) and parallel to the line $x = 1 + 3t$, $y = -1 - 2t$, $z = 3 + t$.

Solution:

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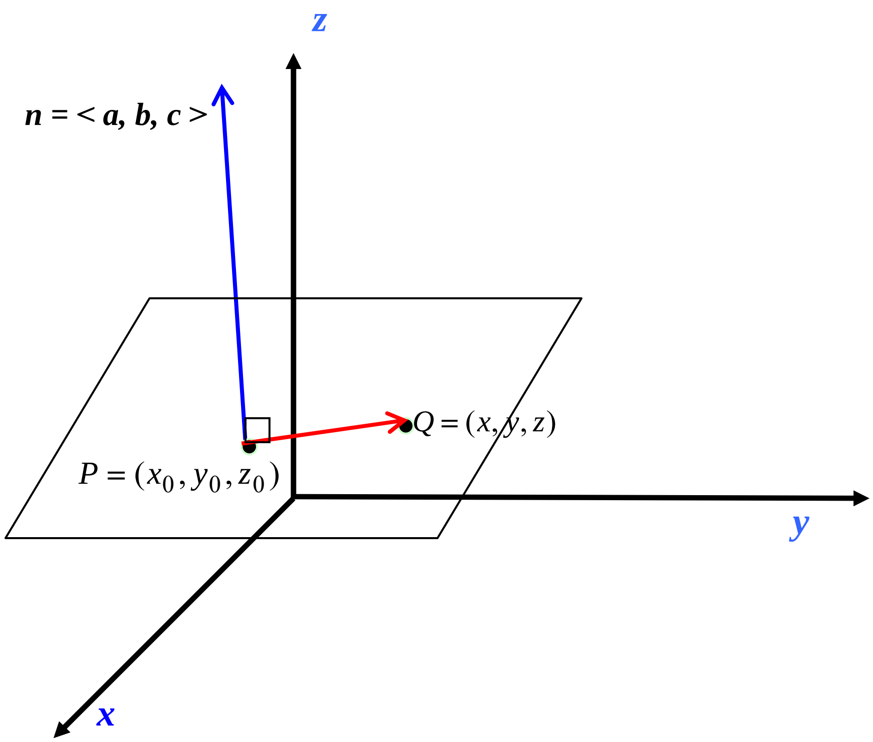
Planes in 3D Space

Consider the plane containing the point $P = (x_0, y_0, z_0)$ and <u>normal</u> <u>vector</u> $\boldsymbol{n} = < a, b, c >$ perpendicular to the plane.



The plane consists of all points $Q = (x, y, z)$ for which the vector $\overrightarrow{\boldsymbol{PQ}}$ is orthogonal to the normal vector $\boldsymbol{n} = < a, b, c >$. Since $\overrightarrow{\boldsymbol{PQ}}$ and $\boldsymbol{n}$ are orthogonal, the following equations hold:

$$\boldsymbol{n} \cdot \overrightarrow{\boldsymbol{PQ}} = 0$$

$$< .\boldsymbol{a}.\boldsymbol{b}.\boldsymbol{c} > \cdot < \boldsymbol{x} - \boldsymbol{x}_0, \boldsymbol{y} - \boldsymbol{y}_0, \boldsymbol{z} - \boldsymbol{z}_0 >= 0$$

$$\boldsymbol{a}(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{b}(\boldsymbol{y} - \boldsymbol{y}_0) + \boldsymbol{c}(\boldsymbol{z} - \boldsymbol{z}_0) = 0$$

This gives the *standard equation of a plane*. If we expand this equation we obtain the following equation:

$$ax + by + cz \underbrace{- ax_0 - by_0 - cz_0}_{\text{Constant } d} = 0$$

Setting $d = -ax_0 - by_0 - cz_0$ gives the *general form of the equation of a plane* in 3D space

$$ax + by + cz + d = 0\,.$$

We summarize these results as follows.

Standard and General Equations of a Plane in the 3D space

The standard equation of a plane in 3D space has the form

$$\boldsymbol{a}(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{b}(\boldsymbol{y} - \boldsymbol{y}_0) + \boldsymbol{c}(\boldsymbol{z} - \boldsymbol{z}_0) = 0$$

where (x_0, y_0, z_0) is a point on the plane and $\boldsymbol{n} = < a, b, c >$ is a vector normal (orthogonal to the plane). If this equation is expanded, we obtain the general equation of a plane of the form

$$ax + by + cz + d = 0$$

Note!! To write the equation of a plane in 3D space, we need a point on the plane and a vector normal (orthogonal) to the plane.

Example 4: Find the equation of the plane through the point (-4, 3, 1) that is perpendicular to the vector $\boldsymbol{a} = -4\,\boldsymbol{i} + 7\,\boldsymbol{j} - 2\,\boldsymbol{k}$.

Solution:

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Example 5: Find the equation of the plane passing through the points (1, 2, -3), (2, 3, 1), and (0, -2, -1).

Solution:

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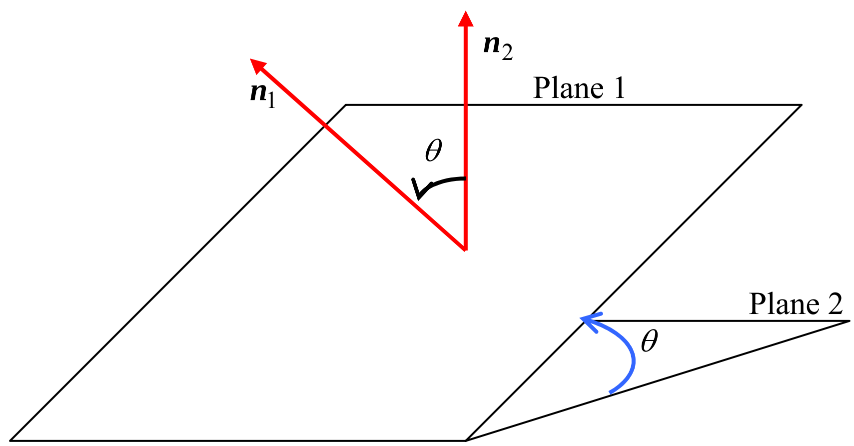
Intersecting Planes

Suppose we are given two intersecting planes with angle θ between them.



Let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be normal vectors to these planes. Then

$$\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$$

Thus, two planes are

1. Perpendicular if $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 0$, which implies $\theta = \frac{\pi}{2}$.

2. Parallel if $\boldsymbol{n}_2 = c\boldsymbol{n}_1$, where c is a scalar.

<u>Notes</u>

1. Given the general equation of a plane $ax + by + cz + d = 0$, the normal vector is $\boldsymbol{n} = < a, b, c >$.
2. The intersection of two planes is a line.

Example 6: Determine whether the planes $3x + y - 4z = 3$ and $-9x - 3y + 12z = 4$ are orthogonal, parallel, or neither. Find the angle of intersection and the set of parametric equations for the line of intersection of the plane.

Solution:

$\blacksquare$

Example 7: Determine whether the planes $x-3y+6z=4$ and $5x+y-z=4$ are orthogonal, parallel, or neither. Find the angle of intersection and the set of parametric equations for the line of intersection of the plane.

Solution: For the plane $x-3y+6z=4$, the normal vector is $\boldsymbol{n}_1 = <1,-3,6>$ and for the plane $5x+y-z=4$, the normal vector is $\boldsymbol{n}_2 = <5,1,-1>$. The two planes will be orthogonal only if their corresponding normal vectors are orthogonal, that is, if $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 0$. However, we see that

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = <1,-3,6> \cdot <5,1,-1> = (1)(5)+(-3)(1)+(6)(-1) = 5-3-6 = -4 \neq 0$$

Hence, the planes are not orthogonal. If the planes are parallel, then their corresponding normal vectors must be parallel. For that to occur, there must exist a scalar *k* where

$$\boldsymbol{n}_2 = k\ \boldsymbol{n}_1$$

Rearranging this equation as $k\boldsymbol{n}_1 = \boldsymbol{n}_2$ and substituting for $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ gives

$$k<1,-3,6> = <5,1,-1>$$

or

$$<k,-3k,6k> = <5,1,-1>.$$

Equating components gives the equations

$$k=5, \quad -3k=1, \quad 6k=-1$$

which gives

$$k=5, \quad k=-\frac{1}{3}, \quad k=-\frac{1}{6}.$$

Since the values of *k* are not the same for each component to make the vector $\boldsymbol{n}_2$ a scalar multiple of the vector $\boldsymbol{n}_1$, the planes are not parallel. Thus, the planes must intersect in a straight line at a given angle. To find this angle, we use the equation

$$\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|}$$

For this formula, we have the following:

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = <1,-3,6> \cdot <5,1,-1> = (1)(5)+(-3)(1)+(6)(-1) = 5-3-6 = -4$$

$$|\boldsymbol{n}_1| = \sqrt{(1)^2+(-3)^2+(6)^2} = \sqrt{1+9+36} = \sqrt{46}$$

$$|\boldsymbol{n}_2| = \sqrt{(5)^2+(1)^2+(-1)^2} = \sqrt{25+1+1} = \sqrt{27}$$

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Thus,

$$\cos\theta = \frac{-4}{\sqrt{46}\ \sqrt{27}}$$

Solving for θ gives

$$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{46}\ \sqrt{27}}\right) \approx 1.68\ \text{radians} \approx 96.5^0 .$$

To find the equation of the line of intersection between the two planes, we need a point on the line and a parallel vector. To find a point on the line, we can consider the case where the line touches the *x*-*y* plane, that is, where *z* = 0. If we take the two equations of the plane

$$\begin{aligned} x - 3y + 6z &= 4 \\ 5x + y - z &= 4 \text{ ‘} \end{aligned}$$

and substitute *z* = 0, we obtain the system of equations

$$x - 3y = 4 \qquad (1)$$
$$5x + y = 4 \qquad (2)$$

Taking the first equation and multiplying by -5 gives

$$\begin{aligned} -5x + 15y &= -20 \\ 5x + y &= 4 \end{aligned}$$

Adding the two equations gives 16*y* = -16 or $y = -\frac{16}{16} = -1$. Substituting $y = -1$ back into equation (1) gives $x - 3(-1) = 4$ or $x + 3 = 4$. Solving for *x* gives *x* = 4-3 = 1. Thus, the point on the plane is (1, -1, 0). To find a parallel vector for the line, we use the fact that since the line is on both planes, it must be orthogonal to both normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. Since the cross product $\boldsymbol{n}_1 \times \boldsymbol{n}_2$ gives a vector orthogonal to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, $\boldsymbol{n}_1 \times \boldsymbol{n}_2$ will be a parallel vector for the line. Thus, we say that

$$\begin{aligned} \boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -3 & 6 \\ 5 & 1 & -1 \end{vmatrix} &= \boldsymbol{i}\begin{vmatrix} -3 & 6 \\ 1 & -1 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 6 \\ 5 & -1 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & -3 \\ 5 & 1 \end{vmatrix} \\ &= \boldsymbol{i}\,(3-6) - \boldsymbol{j}(-1-30) + \boldsymbol{k}(1 - -15) \\ &= -3\boldsymbol{i} + 31\boldsymbol{j} + 16\boldsymbol{k} \end{aligned}$$

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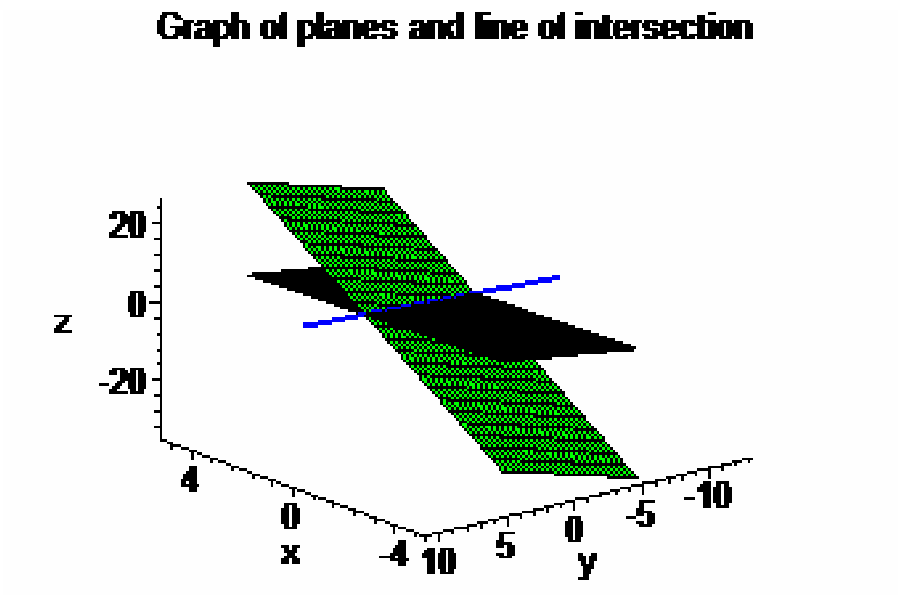
Hence, using the point (1, -1, 0) and the parallel vector $\boldsymbol{v} = -3\boldsymbol{i} + 31\boldsymbol{j} + 16\boldsymbol{k}$, we find the parametric equations of the line are

$$x = 1 - 3t, \quad y = -1 + 31t, \quad z = 16t$$

The following shows a graph of the two planes and the line we have found.



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Example 8: Find the point where the line *x* = 1 + *t*, *y* = 2*t*, and *z* = -3*t* intersects the plane $-4x + 2y - 4z = -2$.

Solution:

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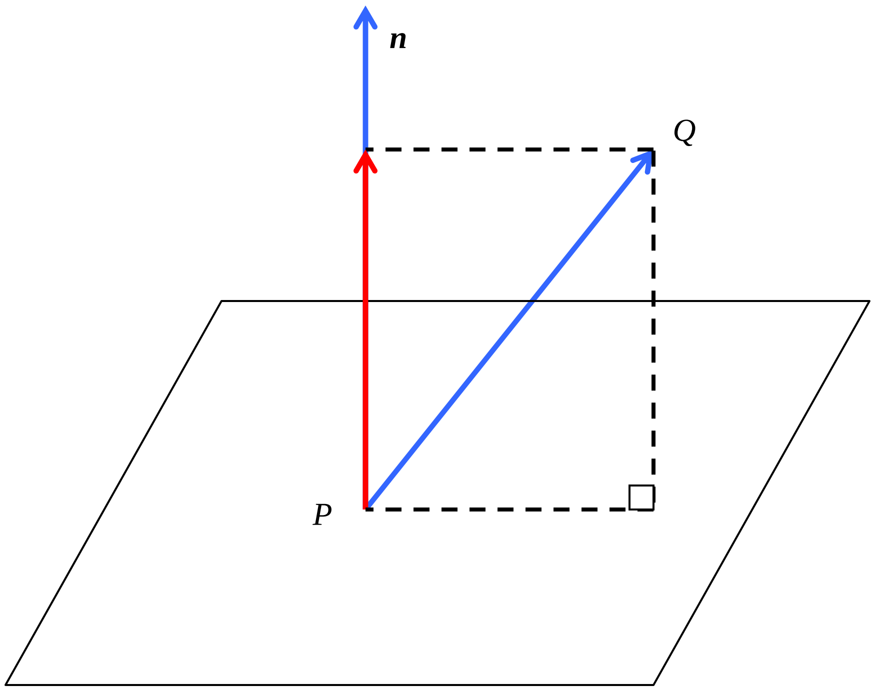
Distance Between Points and a Plane

Suppose we are given a point Q not in a plane and a point P on the plane and our goal is to find the shortest distance between the point Q and the plane.



By projecting the vector $\overrightarrow{\boldsymbol{PQ}}$ onto the normal vector $\boldsymbol{n}$ (calculating the scalar projection $\boldsymbol{comp}_n \overrightarrow{\boldsymbol{PQ}}$), we can find the distance D.

$$\begin{array}{c}\text{Distance Between } \boldsymbol{Q}\\ \text{and the plane}\end{array} = \mathrm{D} = |\boldsymbol{comp}_n \overrightarrow{\boldsymbol{PQ}}| = \frac{|\overrightarrow{\boldsymbol{PQ}} \cdot \boldsymbol{n}|}{|\boldsymbol{n}|}$$

Example 9: Find the distance between the point (1, 2, 3) and line $2x - y + z = 4$.

Solution: Since we are given the point Q = (1, 2, 3), we need to find a point on the plane $2x - y + z = 4$ in order to find the vector $\overrightarrow{\boldsymbol{PQ}}$. We can simply do this by setting $y = 0$ and $z = 0$ in the plane equation and solving for x. Thus we have

$$\begin{aligned} 2x - y + z &= 4 \\ 2x - 0 + 0 &= 4 \\ 2x &= 4 \\ x &= \frac{4}{2} = 2 \end{aligned}$$

Thus P = (2, 0, 0) and the vector $\overrightarrow{\boldsymbol{PQ}}$ is

$$\overrightarrow{\boldsymbol{PQ}} = < 1-2, 2-0, 3-0 > = < -1, 2, 3 > .$$

Hence, using the fact that the normal vector for the plane is $\boldsymbol{n} = < 2, -1, 1 >$, we have

$$\begin{matrix}\text{Distance Between} \\ \text{Q and the plane}\end{matrix} = \frac{|\overrightarrow{\boldsymbol{PQ}} \cdot \boldsymbol{n}|}{|\boldsymbol{n}|} = \frac{|< -1,2,3 > \cdot < 2,-1,1 >}{\sqrt{(2)^2 + (-1)^2 + (1)^2}} = \frac{|-2-2+3|}{\sqrt{4+1+1}} = \frac{|-1|}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

Thus, the distance is $\frac{1}{\sqrt{6}}$.

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