Section 8.5: Power Series

Practice HW from Stewart Textbook (not to hand in)
p. 598 # 3-17 odd

Power Series

Definition: A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n\, x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$$

or more generally, a power series centered at a constant a is given by

$$\sum_{n=0}^{\infty} c_n\,(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots + c_n(x-a)^n + \ldots$$

Recall that n factorial is represented by $n!$ and is given by

$$n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

<u>Examples of Power Series</u>

1. We will soon learn that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

   is a power series centered at 0 that can be used to represent e^x exactly.

2. $\sum_{n=0}^{\infty} n^2 x^n$ is a power series centered at 0.

3. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)\,3^n}$ is a power series centered at 2.

Convergence of a Power Series

For a power series centered at *a*, precisely one of the following is true.
1. The series converges only at *a*.
2. There exist a real number $R > 0$ such that the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$.
3. The series converges for all real numbers.

Notes
1. *R* is called the radius of convergence. The set of all values of *x* for which the power series converges is called the interval of convergence.
2. $|x-a| < R$ represents the values of *x* whose distance from *a* is smaller than the radius of convergence *R*. In interval notation, means $-R < x-a < R$ or $-R+a < x < R+a$.
3. A useful property for the absolute value is that for any two numbers *a* and *b*, $|ab| = |a||b|$.
4. To determine the initial interval radius of convergence, we use the ratio test (see below).
5. It may be possible to extend the interval of convergence to the endpoints of the interval found by the ratio test. This involves using the series convergence tests studied in previous sections.

Ratio Test

Given an infinite series $\sum a_n$.

1. $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. $\sum a_n$ diverges if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$.
3. The ratio test is inconclusive if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Example 1: Determine the interval of convergence and radius of convergence for the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution:

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Example 2: Determine the interval of convergence and radius of convergence for the series $\sum_{n=0}^{\infty} n^2 x^n$.

Solution:

Example 3: Determine the interval of convergence and radius of convergence for the series $\sum_{n=0}^{\infty} n!\,x^n$.

Solution:

Example 4: Determine the interval of convergence and radius of convergence for the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)\,3^n}$.

Solution: To set up the ratio test, we start by letting $a_n = \frac{(x-2)^n}{(n+1)\,3^n}$. Replacing every occurrence of n with n + 1 gives $a_{n+1} = \frac{(x-2)^{n+1}}{((n+1)+1)\,3^{n+1}} = \frac{(x-2)^{n+1}}{(n+2)\,3^{n+1}}$. We then determine the interval of convergence using the ratio test as follows:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(x-2)^{n+1}}{(n+2)\,3^{n+1}}}{\dfrac{(x-2)^n}{(n+1)\,3^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+2)\,3^{n+1}}\,\frac{(n+1)\,3^n}{(x\text{ - }2)^n}\right| \qquad \text{(Take the reciprocal of the denominator and multiply)}$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^n\,(x-2)\,(n+1)\,3^n}{(n+2)\,3^n\,3^1(x\text{ - }2)^n}\right| \qquad \left(\begin{array}{c}\text{Use property of exponents } b^{x+y} = b^x b^y \\ (x-2)^{n+1} = (x-2)^n\,(x-2),\ \ 3^{n+1} = 3^n\,3^1\end{array}\right)$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(x-2)^n}\,(x-2)\,(n+1)\cancel{3^n}}{(n+2)\cancel{3^n}\,3^1\cancel{(x\text{ - }2)^n}}\right| \qquad \text{(Cancel common terms in numer and denom)}$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)\,(n+1)}{3\,(n+2)}\right| \qquad \text{(Simplify)}$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)}{3}\right|\,\left|\frac{n+1}{n+2}\right| \qquad \text{(Use multiplication property of limits)}$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)}{3}\right|\,\lim_{n\to\infty}\frac{n+1}{n+2} \qquad \left(\begin{array}{c}\text{Break limit into terms involving } x \text{ and } n \\ \text{Since } n\to\infty, n>0 \text{ and absolute value can be dropped}\end{array}\right)$$

$$= \lim_{n\to\infty}\frac{|\,x-2\}}{3}\ \ (1) \qquad \text{(Use L'Hopital's rule to evaluate limit involving } n$$

$$= \frac{|\,x-2\}}{3} \qquad \left(\begin{array}{c}\text{Since limit involves variable } n, \text{ limit of terms} \\ \text{involving } x \text{ is treated as constants}\end{array}\right)$$

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For a power series to converge, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$. Hence, this means that from the result on the previous page, $\frac{|x-2|}{3}<1$ or $|x-2|<3$. Thus, the radius of convergence is $\boxed{R=3}$. Since $|x-2|<3$, by the definition of absolute value, $-3<x-2<3$. Adding 2 to each side of the inequality gives the interval of convergence for the series, $-1<x<5$. We can possibly increase the size of the interval of convergence by testing the endpoints, that is, by testing whether the given series $\sum_{n=0}^{\infty}\frac{(x-2)^n}{(n+1)\,3^n}$ is convergent at x = -1 and x = 5. We test these endpoints as follows:

$$x=-1 \Rightarrow \sum_{n=0}^{\infty}\frac{(-1-2)^n}{(n+1)\,3^n}=\sum_{n=0}^{\infty}\frac{(-3)^n}{(n+1)\,3^n}$$

$$=\sum_{n=0}^{\infty}\frac{(-1\cdot 3)^n}{(n+1)\,3^n}$$

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$$=\sum_{n=0}^{\infty}\frac{(-1)^n\,3^n}{(n+1)\,3^n} \qquad \left(\begin{array}{c}\text{Use law of exponents } (ab)^n=a^nb^n \\ (-1\cdot 3)^n=(-1)^n\,3^n\end{array}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\cancel{3^n}}{(n+1)\cancel{3^n}} \qquad (\text{Simplify})$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)}$$

We can, by the alternating series test, show that the $\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)}$ series is convergent. Note that this is true since for $a_n=\frac{1}{n+1}$, generating terms of the sequence we see that

1. $\frac{1}{2}>\frac{1}{3}>\frac{1}{4}>\frac{1}{5}>\ldots$
2. $\lim_{n\to\infty}\frac{1}{n+1}=0$.

Thus the series $\sum_{n=0}^{\infty}\frac{(x-2)^n}{(n+1)\,3^n}$ is convergent at x = -1.

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For the endpoint $x = 5$

$$x = 5 \Rightarrow \sum_{n=0}^{\infty} \frac{(5-2)^n}{(n+1)\,3^n} = \sum_{n=0}^{\infty} \frac{\cancel{3^n}}{(n+1)\,\cancel{3^n}} \qquad \text{(Simplify)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)}$$

The series $\sum_{n=0}^{\infty} \frac{1}{(n+1)}$ can be shown to be divergent by the limit comparison test. The series we will compare this series with is $\sum_{n=0}^{\infty} \frac{1}{n}$. Then, setting $a_n = \frac{1}{n+1}$ and $b_n = \frac{1}{n}$, we have

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} \qquad \text{(Take the reciprocal of denominator and multiply)}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1} \qquad \text{(Apply L'Hopital's rule for the indeterminate form } \tfrac{\infty}{\infty}\text{)}$$

$$= 1 > 0$$

Thus, the limit comparison test will apply. Since the series $\sum_{n=0}^{\infty} \frac{1}{n}$,where $p = 1 \le 1$, is a divergent series, the series $\sum_{n=0}^{\infty} \frac{1}{(n+1)}$ diverges. Hence, the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)\,3^n}$ is divergent at $x = 5$.

Summarizing, the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)\,3^n}$ is convergent at $x = -1$, $-1 < x < 5$, and divergent at $x = 5$. Thus the interval of convergence is $\boxed{-1 \le x < 5 \text{ or } [-1, 5)}$ in interval notation.

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