Section 8.3: The Integral and Comparison Tests; Estimating Sums

Practice HW from Stewart Textbook (not to hand in)
p. 585 # 3, 6-12, 13-25 odd

In this section, we want to determine other methods for determining whether a series converges or diverges.

The Integral Test

For a function *f*, if $f(x) > 0$, is continuous and decreasing for $x \geq M$ and $a_n = f(n)$, then

$$\sum_{n=M}^{\infty} a_n \quad \text{and} \quad \int_M^{\infty} f(x)\,dx$$

either both converge or both diverge.

Note: The integral test is only a test for convergence or divergence. In the case of convergence, it does not find a value for the sum of the series.

Example 1: Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{4n+1}$$

Solution:

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Example 2: Determine the convergence or divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Solution: We start by writing the formula for the sequence as a function of *x*, that is, we write $a_n = \frac{1}{n(\ln n)^2}$ as $f(x) = \frac{1}{x(\ln x)^2}$. We should note first of all that for $x > 2$,

1. $f(x) = \frac{1}{x(\ln x)^2}$ is always positive (> 0), **2.** continuous (the function is only undefined when $x \le 0$ and when $x = 1$ since ln 1 = 0), and decreasing), and **3.** decreasing (as $x \to \infty$, $f(x) = \frac{1}{x(\ln x)^2} \to 0$. The following graph of this function generated using Maple should help convince you of these facts:

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> f := x -> 1/(x*ln(x)^2);
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$$f := x \to \frac{1}{x \ln(x)^2}$$

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> plot(f(x), x = 2..10, thickness = 2, view = [-1..10,
  -2..2], title = "Graph of f(x) = 1/(x*ln(x)^2");
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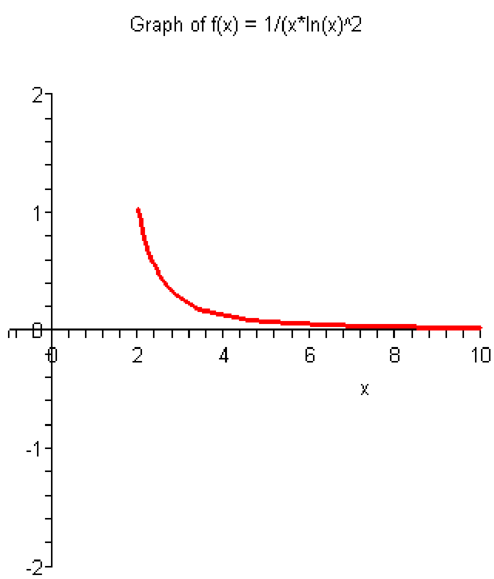

Thus, the integral test can be applied. We first set up the improper integral of the function and integrate as follows:

(continued on next page)

Note to integrate, use $u - du$ subs

$$\int_2^{\infty} \frac{1}{x\,(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x\,(\ln x)^2}\,dx$$

Let $u = \ln x$, $\quad du = \frac{1}{x}dx$

$$\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

$$= \lim_{t\to\infty}\left[-\frac{1}{\ln x}\Big|_2^t \right] \qquad \text{(Result of integral)}$$

$$= \lim_{t\to\infty}\left[-\frac{1}{\ln t} - -\frac{1}{\ln 2} \right] \qquad \text{(Substitute the limits)}$$

$$= -0 + \frac{1}{\ln 2} \qquad \text{(Evaluate the limit as } t \to \infty, \frac{1}{\ln t} \to 0\text{)}$$

$$= \frac{1}{\ln 2}$$

Since the improper integral evaluates to a fixed number $(1/\ln 2)$, it is convergent. Thus by the integral test, the series $\boxed{\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^2} \text{ is convergent.}}$

$\blacksquare$

Example 3: Show why the integral test cannot be used to analyze the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Solution:

$\blacksquare$

p-Series and Harmonic Series

A p-series series is given by

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$

If $p = 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \text{ is called a \textit{harmonic} series.}$$

Convergence of p – series

A p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\dots$$

1. Converges if $p > 1$.
2. Diverges if $p \le 1$.

Example 4: Determine whether the p-series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\dots$ is convergent or divergent.

Solution:

$\blacksquare$

Example 5: Determine whether the p-series $\sum_{n=1}^{\infty}\frac{1}{n}$ is convergent or divergent.

Solution:

$\blacksquare$

Example 6: Determine whether the p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is convergent or divergent.

Solution:

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Making Comparisons between Series that are Similar

Many times we can determine the convergence or divergence of a series by comparing it with the known convergence or divergence of a related series. For example,

$\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ is close to the *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$,

$\sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$ is close to the geometric series $\sum_{n=1}^{\infty} \frac{4^n}{3^n}$.

Under the proper conditions, we can use a series where it is easy to determine the convergence or divergence and use it to determine convergence or divergence of a similar series using types of comparison tests. We will examine two of these tests – the *direct comparison* test and the *limit comparison* test.

Direct Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series only with positive terms ($a_n > 0$ and $b_n > 0$).

1. If $\sum b_n$ is convergent and $a_n \le b_n$ for all terms n, $\sum a_n$ is convergent.
2. If $\sum b_n$ is divergent and $a_n \ge b_n$ for all terms n, $\sum a_n$ is divergent.

Note: Most of the time, we will compare the given series to a p-series or a geometric series.

Example 7: Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$ is convergent or divergent.

Solution:

Example 8: Determine whether the series $\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$ is convergent or divergent.

Solution:

$\blacksquare$

Example 9: Demonstrate why the direct comparison test cannot be used to analyze the convergence or divergence of the series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n^2 + 1}$

Solution:

$\blacksquare$

Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series only with <u>positive</u> terms ($a_n > 0$ and $b_n > 0$) and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \quad \text{where } L \text{ is a finite number and } L > 0.$$

Then either $\sum a_n$ and $\sum b_n$ either both converge or $\sum a_n$ and $\sum b_n$ both diverge.

Note: This test is useful when comparing with a p-series. To get the p-series to compare with take the highest power of the numerator and simplify.

Example 10: Determine whether the series $\sum_{n=1}^{\infty} \frac{5n-3}{n^2 - 2n + 5}$ is convergent or divergent.

Solution: