Section 8.2: Series

Practice HW from Stewart Textbook (not to hand in)
p. 575 # 9-15 odd, 19, 21, 23, 25, 31, 33

Infinite Series

Given an infinite sequence $\{a_n\}$, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

is called an *infinite series*.

Note: $a_n = \dfrac{n}{n+1}$ is the infinite sequence $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$

$\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots$ is an infinite series.

Consider

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \ldots + a_n
\end{aligned}$$

Note that $S_1, S_2, \ldots, S_n$ is a <u>sequence</u> of numbers called a *sequence of partial sums*.

Definition: For an infinite series $\sum a_n$, the n^{th} partial sum is given by

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$

If the sequence of partial sums $\{S_n\}$ converges to S, the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is the sum of the series

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} (a_1 + a_2 + \ldots + a_n) = \sum_{n=1}^{\infty} a_n .$$

If the sequence $\{S_n\}$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 1: Find the first five sequence of partial sum terms of the series

$$\sum_{n=1}^{\infty} n$$

Find a formula that describes the sequence of partial sums and determine whether the sequence converges or diverges.

Solution:

∎

Example 2: Find the first five sequence of partial sum terms of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

Find a formula that describes the sequence of partial sums and determine whether the sequence converges or diverges.

Solution:

Geometric Series

A geometric series is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots + ar^n + \ldots$$

with ratio *r*.

Notes

1. The geometric series <u>converges</u> if and only if $|r| < 1$. When $|r| < 1$, the sum of the series (the value the series converges to) is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

   If $|r| \geq 1$, then the geometric series diverges.

2. The value a is the <u>first</u> <u>term</u> of the series.

3. The ratio r is the factor you multiply the previous term by to get the next one. That is,

$$r\,(n^{th}\text{ term}) = (n+1)^{th}\text{ term} \qquad \text{or} \qquad r = \frac{n^{th}\text{ term}}{(n+1)^{th}\text{ term}}$$

Example 3: Determine whether the series $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}$ is convergent or divergent. If convergent, find its sum.

Solution:

$\blacksquare$

Example 4: Determine whether the series $3-1+\frac{1}{3}-\frac{1}{9}+\ldots$ is convergent or divergent. If convergent, find its sum.

Solution:

$\blacksquare$

Example 5: Determine whether the series $\sum_{n=1}^{\infty} 2^{-n} 5^{n+1}$ is convergent or divergent. If convergent, find its sum.

Solution:

$\blacksquare$

<u>Properties</u> <u>of</u> <u>Series</u> (p. 573) If $\sum a_n$ and $\sum b_n$ are convergent series, then

1. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n,$ $\quad$ c is a constant.

2. $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

3. $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 6: Determine whether the series $\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}+\frac{2}{3^{n-1}}\right)$ is convergent or divergent. If convergent, find its sum.

Solution:

$\blacksquare$

Applications of Geometric Series

Example 7: Express $0.\overline{73} = 0.73737373\ldots$ as a ratio of integers.

Solution: Note that we can write the given number as

$$0.73737373\ldots = 0.73 + 0.0073 + 0.000073 + 0.00000073 + \ldots$$

This is a geometric series with $r = \frac{1}{100} = 0.01$. Note that $|r| = |0.01| = 0.01 < 1$. Also, for this series, a = 0.73. Thus, the number can be expressed as the following ratio.

$$0.73737373\ldots = \frac{a}{1-r} = \frac{0.73}{1-0.01} = \frac{0.73}{0.99} = \frac{\frac{73}{100}}{\frac{99}{100}} = \frac{73}{\cancel{100}} \cdot \frac{\cancel{100}}{99} = \boxed{\frac{73}{99}}$$

$\blacksquare$

Tests For Non-Geometric Series

Most series are <u>not</u> <u>geometric</u> – that is, there is not a ratio r that you multiply each term to get to the next term. We will be looking at other ways to determine the convergence and divergence of series in upcoming sections.

<u>Some other ways to test series</u>

1. Divergence Test: If the sequence $\{a_n\}$ <u>does</u> <u>not</u> <u>converge</u> <u>to</u> <u>0</u>, then the series $\sum a_n$ <u>diverges</u>. Note: This is only a test for <u>divergence</u> – if the sequence $\{a_n\}$ converges to 0 does not necessarily mean the series $\sum a_n$ converges.
2. Examine the partial sums to determine convergence or divergence (Examples 1 and 2 of this section)
3. Techniques discussed in upcoming sections.

Example 8: Demonstrate why the series $\sum_{n=1}^{\infty} \frac{n}{2n+3} = \frac{1}{5} + \frac{2}{7} + \frac{3}{9} + \frac{4}{11} + \dots$ is not geometric. Then analyze whether the series is convergent or divergent.

Solution:

$\blacksquare$

Example 9: Analyze whether the series $\sum_{n=1}^{\infty} \frac{1}{4n+1}$ is convergent or divergent.

Solution:

$\blacksquare$