Section 8.1: Sequences

Practice HW from Stewart Textbook (not to hand in)
p. 565 # 3-33 odd

Sequences

Sequences are collection of numbers or objects that is ordered by the positive integers.

Notation: $a_1, a_2, a_3, a_4, \ldots, a_n$ (known as the terms of the sequence).

Example 1: Write the first five terms of the sequence $a_n = \dfrac{n}{n+1}$.

Solution:

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Example 2: Write the first five terms of the sequence $a_n = \frac{(-1)^n}{3^n}$.

Solution:

Describing the nth term of a sequence

Involves writing a formula describing the pattern the sequence of numbers follows.

Example 3: Find a formula for the general term a_n of the sequence $\{3, 7, 11, 15, \ldots\}$.

Solution:

Example 4: Find a formula for the general term a_n of the sequence $\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots\}$.

Solution:

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Convergence of Sequences

Sequences whose behavior approaches a limiting value are said to <u>converge</u> to this value

Example 5: What value does the sequence $\left\{\frac{1}{2^n}\right\}$ appear to converge to?

Solution:

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Definition: A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n \to \infty} a_n = L$$

which is read as a_n approaches L as n approaches ∞ if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, then the sequence is convergent. Otherwise, it is divergent.

Useful Facts for Showing Convergence of Sequences

1. L Hopital's Rule for indeterminate forms $\frac{\infty}{\infty}$.
2. Absolute Value Theorem: If $\lim |a_n| = 0$, then $\lim a_n = 0$.
3. Using Maple to graph.

Example 6: Determine whether the sequence $a_n = n^2 + n$ converges or diverges. If the sequence converges, find the limit.

Solution:

Example 7: Determine whether the sequence $a_n = \dfrac{n^2+1}{3n^2-1}$ converges or diverges. If the sequence converges, find the limit.

Solution:

∎

Example 8: Determine whether the sequence $a_n = \dfrac{e^n}{e^{2n}-1}$ converges or diverges. If the sequence converges, find the limit.

Solution:

Example 9: Determine whether the sequence $a_n = \cos n\pi$ converges or diverges. If the sequence converges, find the limit.

Solution:

Example 10: Determine whether the sequence $a_n = \ln(2n+1) - \ln(n)$ converges or diverges. If the sequence converges, find the limit.

Solution: Note if we compute $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln(2n+1) - \ln(n)$ directly, we obtain $\infty - \infty$, which is an indeterminant form (this is not necessarily zero!). However, we can evaluate this limit by rewriting the sequence formula using some basic algebra. We have

$$\lim_{n\to\infty} \ln(2n+1) - \ln(n) = \lim_{n\to\infty} \ln\left(\frac{2n+1}{n}\right) \qquad (\text{Use ln property } \ln u - \ln v = \ln\frac{u}{v})$$

$$= \lim_{n\to\infty} \ln\left(\frac{2n}{n} + \frac{1}{n}\right) \qquad (\text{Use property of fractions } \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c})$$

$$= \lim_{n\to\infty} \ln\left(2 + \frac{1}{n}\right) \qquad (\text{Simplify})$$

$$= \ln(2+0) \qquad (\text{Evaluate limit, as } n \to \infty, \frac{1}{n} \to 0)$$

$$\boxed{= \ln 2}$$

Thus, the sequence $a_n = \ln(2n+1) - \ln(n)$ converges to ln 2.

Example 11: Determine whether the sequence $a_n = (-1)^n \dfrac{n}{n^2+1}$ converges or diverges. If the sequence converges, find the limit.

Solution:

Factorial

Recall that $n!$, read as *n* factorial, is defined to be

$n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$.

Note: $0! = 1$.

Example 12: Compute $5!$.

Solution:

Example 13: Determine whether the sequence $a_n = \frac{e^n}{n!}$ converges or diverges. If the sequence converges, find the limit.

Solution: Taking the $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n!}$ gives the indeterminant form $\frac{\infty}{\infty}$. However, using L' Hopital's rule to find the limit is not practical since taking the derivative of *n*! is not trivial. However, Maple can be used to easily plot the behavior of the terms in this sequence. This can be accomplished with the following two commands:

```
> f := n -> exp(n)/factorial(n);
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$$f := n \rightarrow \frac{e^n}{n!}$$

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> plot([seq([i, f(i)], i = 1..20)], style = point, view =
[-2..20, -2..10], thickness = 2, title = "Graph of sequence
e^n/n!");
```

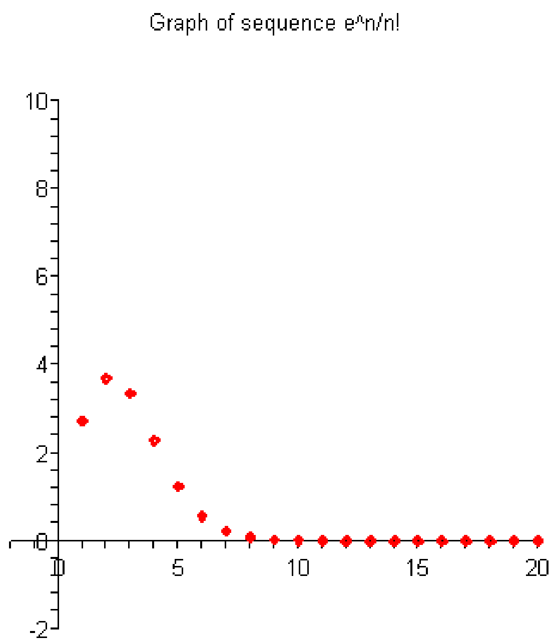

Thus, it appears that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n!} = 0$. Hence, the sequence most likely converges to 0.