Section 7.4: Exponential Growth and Decay

Practice HW from Stewart Textbook (not to hand in)
p. 532 # 1-17 odd

In the next two sections, we examine how population growth can be modeled using differential equations. We start with the basic exponential growth and decay models. Before showing how these models are set up, it is good to recall some basic background ideas from algebra and calculus.

1. A variable y is *proportional* to a variable x if $y = k\,x$, where k is a constant.
2. Given a function $P(t)$, where P is a function of the time t, the rate of change of P with respect to the time t is given by $\dfrac{dP}{dt} = P'(t)$.
3. A function $P(t)$ is increasing over an interval if $\dfrac{dP}{dt} = P'(t) > 0$.

   A function $P(t)$ is decreasing over an interval if $\dfrac{dP}{dt} = P'(t) < 0$.

   A function $P(t)$ is neither increasing or decreasing over an interval if $\dfrac{dP}{dt} = P'(t) = 0$.

The Exponential Growth Model

When a population grows exponentially, it grows at a rate that is proportional to its size at any time t. Suppose the variable $P(t)$ (sometimes we use just use P) represents the population at any time t. In addition, let P_0 be the initial population at time $t = 0$, that is, $P(0) = P_0$. Then if the population grows exponentially,

(Rate of change of population at time t) = k (Current population at time t)

In mathematical terms, this can be written as

$$\frac{dP}{dt} = kP.$$

Solving for k gives

$$k = \frac{1}{P}\frac{dP}{dt}$$

The value k is known as the *relative growth rate* and is a constant.

Suppose we return to the equation

$$\frac{dP}{dt} = kP\,.$$

We can solve this equation using separation of variables. That is,

$$\frac{dP}{P} = kdt \qquad \text{(Separate the variables)}$$

$$\int \frac{1}{P}\,dP = \int kdt \qquad \text{(Integrate both sides)}$$

$$\ln|P| = kt + C \qquad \text{(Apply integration formulas)}$$

$$e^{\ln|P|} = e^{kt+C} \qquad \text{(Raise both sides to exponential function of base } e\text{)}$$

$$|P| = e^{kt}e^{C} \qquad \text{(Use inverse property } e^{\ln k} = k \text{ and law of exponents } b^{x+y} = b^x b^y\text{)}$$

$$P(t) = Ae^{kt} \qquad \text{(Use absolute value definition } P = \pm e^{C}e^{kt} \text{ and replace constant } \pm e^{C} \text{ with } A.\text{)}$$

The equation $P(t) = Ae^{kt}$ represents the general solution of the differential equation. Using the initial condition $P(0) = P_0$, we can find the particular solution.

$$P_0 = P(0) = Ae^{k(0)} \qquad \text{(Substitute } t = 0 \text{ in the equation and equate to } P_0\text{)}$$

$$P_0 = A(1) \qquad \text{(Note that } e^{k(0)} = e^{0} = 1\text{)}$$

$$A = P_0 \qquad \text{(Solve for } A\text{)}$$

Hence, $P(t) = P_0 e^{kt}$ is the particular solution. Summarizing, we have the following:

<u>Exponential Growth Model</u>

The initial value problem for exponential growth

$$\frac{dP}{dt} = kP,\ P(0) = P_0$$

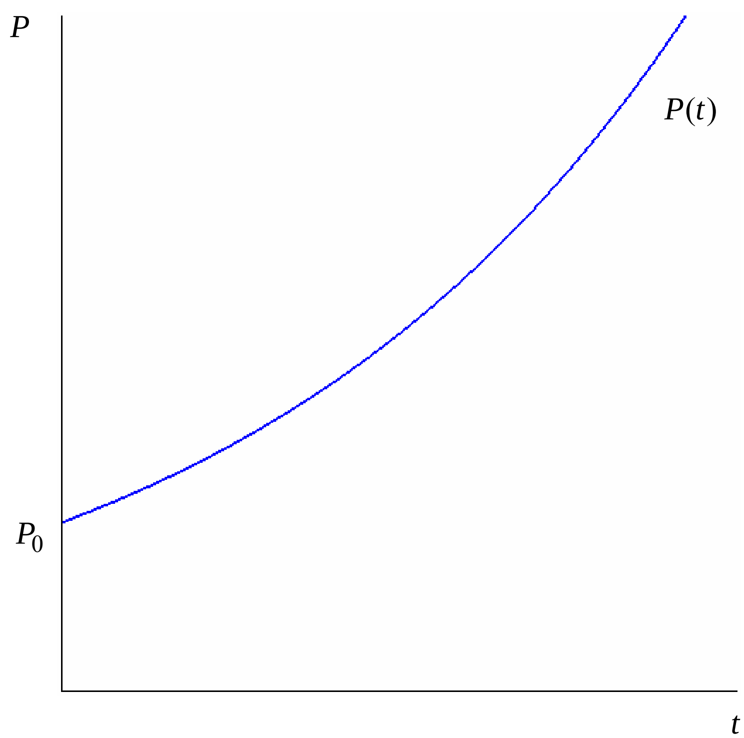
has particular solution

$$P(t) = P_0 e^{kt}$$

where P_0 = initial population (population you that with) at time $t = 0$,
k = relative growth rate that is constant
t = the time the population grows.
$P(t)$ = what the population grows to after time t.

Notes[

1. When modeling a population with an exponential growth model, if the relative growth rate k is unknown, it should be determined. This is usually done using the known population at two particular times.
2. Exponential growth models are good predictors for small populations in large populations with abundant resources, usually for relatively <u>short</u> <u>time</u> <u>periods</u>.
3. The graph of the exponential equation $P(t) = P_0 e^{kt}$ has the general form



Example 1: Solve a certain organism develops with a constant relative growth of 0.2554 per member per day. Suppose the organism starts on day zero with 10 members. Find the population size after 7 days.

Solution:

Example 2: A population of a small city had 3000 people in the year 2000 and has grown at a rate proportional to its size. In the year 2005 the population was 3700.

a. Find an expression for the number of people in the city t years after the year 2000.
b. Estimate the population of the city in 2006. In 2010.
c. Find the rate of growth of the population in 2006.
d. Assuming the growth continues at the same rate, when will the town have 25000 people?

Solution:

Exponential Decay

When a population decays exponentially, it decreases at a rate that is proportional to its size at any time *t*. The model for exponential decay is

$$\frac{dP}{dt} = -kP, \quad P(t) = P_0$$

Here, $k = -\frac{1}{P}\frac{dP}{dt}$ is called the *relative decay constant*. Note that $k > 0$ since, because the population is decreasing, $\frac{dP}{dt} < 0$ and $k = \underbrace{-\frac{1}{P}}_{negative} \cdot \underbrace{\frac{dP}{dt}}_{negative} > 0$. Using separation of variables in a process similar to exponential growth, it can be shown that the solution to the initial value problem is $P(t) = P_0 e^{-kt}$. Summarizing, we have the following:

<u>Exponential Decay Model</u>

The initial value problem for exponential decay

$$\frac{dP}{dt} = -kP, \; P(0) = P_0$$

has particular solution

$$P(t) = P_0 e^{-kt}$$

where P_0 = initial population (population you that with) at time $t = 0$,
k = relative decay rate that is constant. Note that $k > 0$.
t = the time the population decays.
$P(t)$ = the population that is <u>left</u> after time *t*.

<u>Notes</u>

1. Many times the rate of decay is expressed in terms of *half-life*, the time it takes for half of any given quantity to decay so that only half of its original amount remains.
2. Radioactive elements typically decay exponentially.

Example 3: Bismuth-210 has a half-life of 5.0 days.

a. Suppose a sample originally has a mass of 800 mg. Find a formula for the mass remaining after *t* days.
b. Find the mass remaining after 30 days.
c. When is the mass reduced to 1 mg.
d. Sketch the graph of the mass function.

Solution: (Part a) Since this is an exponential decay problem, we will use the formula $P(t) = P_0 e^{-kt}$. Since we start with 800 mg, then we know that $P_0 = 800$. Thus the formula becomes

$$P(t) = 800e^{-kt}$$

To complete the equation that models this population, we need to find the relative decay rate *k*. We can use the half life of the substance to do this. The half life of Bismuth-210 is 5 days. This says that after *t* = 5, the original population of 800 mg has decay to half of its original amount, or $\frac{1}{2}(800) = 400$ mg. Mathematically, since $P(t)$ represents that amount of population of the substance left after time *t*, this says that $P(5) = 400$. Using the decay equation, we have

$$400 = P(5) = 800e^{-k(5)}$$

or rearranging, we have

$$800e^{-5k} = 400$$

We must solve this equation for *k*. We proceed with the following steps.

$$\begin{aligned}
e^{-5k} &= \frac{400}{800} && \text{(Divide by sides by 800)} \\
e^{-5k} &= 0.5 && \text{(Simplify)} \\
\ln e^{-5k} &= \ln(0.5) && \text{(Take ln of both sides)} \\
-5k \ln e &= \ln(0.5) && \text{(Use property } \ln b^u = u \ln b\text{)} \\
-5k(1) &= \ln(0.5) && \text{(Recall } \ln e = 1\text{)} \\
k &= \frac{\ln(0.5)}{-5} && \text{(Divide both sides of - 5)} \\
k &\approx 0.1386 && \text{(Use calculator and round to 4 decimail places)}
\end{aligned}$$

Substituting $k = 0.1386$ and $P_0 = 800$ gives a formula for finding the remaining mass.

$$\boxed{P(t) = 800e^{-0.1386t}}$$

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Part b.) Using the formula $P(t) = 800e^{-0.1386t}$ found in part a, we see that

$$\begin{array}{l}\text{Mass remaining} \\ \text{after } t = 30 \text{ days}\end{array} = P(30) = 800e^{-0.1386(30)} = 800e^{-4.158} \boxed{\approx 12.5 \text{ grams}}$$

Part c.) In this problem, we want the time *t* it takes for the mass to have reduced down to 1 mg. That is, we want *t* when $P(t) = 1$. We perform the following steps using $P(t) = 800e^{-0.1386t}$ to solve for *t*.

$$\begin{aligned}
1 = P(t) &= 800e^{-0.1386t} && \text{(Set } P(t) = 1\text{)} \\
800e^{-0.1386t} &= 1 && \text{(Rearrange the equation)} \\
e^{-0.1386t} &= \frac{1}{800} && \text{(Divide both sides by 800)} \\
\ln e^{-0.1386t} &= \ln\left(\frac{1}{800}\right) && \text{(Take ln of both sides)} \\
-0.1386t \ln e &= \ln\left(\frac{1}{800}\right) && (\ln b^u = u \ln b) \\
-0.1386t \ (1) &= \ln\left(\frac{1}{800}\right) && (\ln e = 1) \\
t &= \frac{\ln(1/800)}{-0.1386} && \text{(Divide both sides by - 0.1386)} \\
t &\approx 48.2 && \text{(Use calculator to approximate)}
\end{aligned}$$

Thus, it takes approximately $\boxed{t = 48.2 \text{ days}}$ for the substance to decay to 1 mg.

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d. The following Maple commands will generate the desired graph:

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> P := 800*exp(-0.1386*t);
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$$P := 800\,\mathrm{e}^{(-0.1386\,t)}$$

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> plot(P, t = 0..60, color = blue, thickness = 2, view = [-
1..60, -10..1000], title = "Graph of 800e^(-0.1386t) for t
= 0..60 for Bismuth-210");
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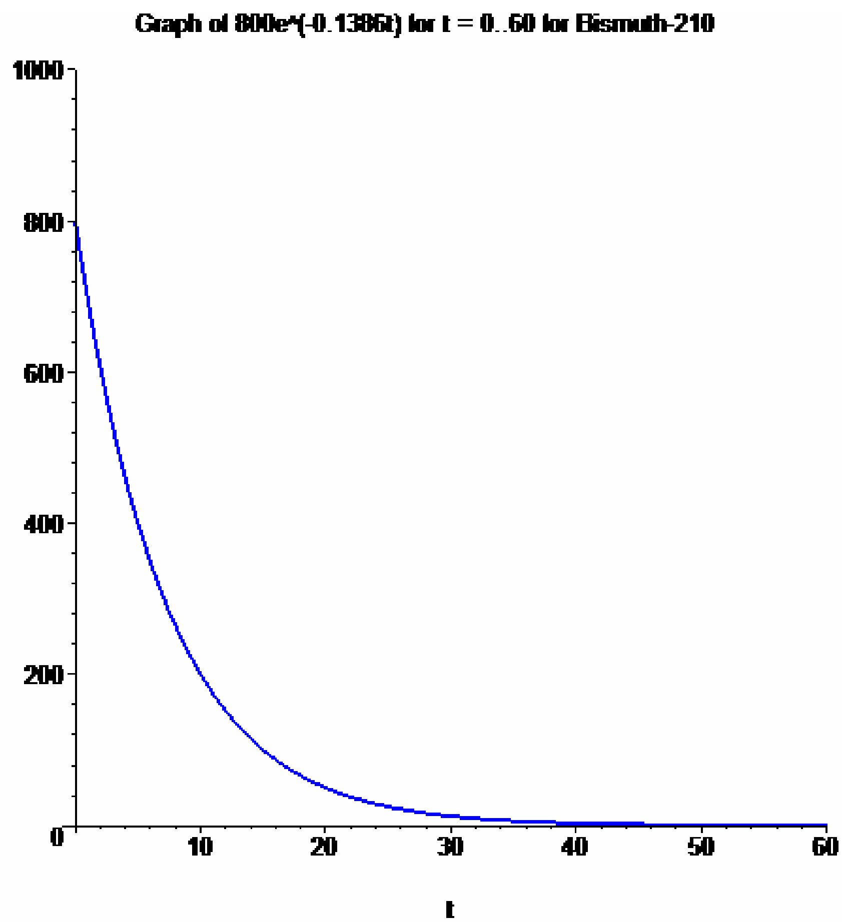

Example 4: *Radiocarbon Dating*. Scientists can determine the age of ancient objects (fossils, for example) using radiocarbon dating. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore, the level of radioactivity must also decay exponentially. Suppose a fossil found has about 35 % as much ^{14}C radioactivity as normal animals do on Earth today. Estimate the age of the fossil.

Solution:

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Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the difference between the object and its surroundings. Let $T(t)$ be the temperature of an object at time *t* and T_s be the temperature of the surroundings (environment). Note that T_s will we be assumed to be constant. Mathematically, Newton's Law of Cooling can be expressed as the following differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

Suppose we let $y = T - T_s$. Then, taking the derivative of both sides with respect to the time *t* gives $\frac{dy}{dt} = \frac{dT}{dt} - 0 = \frac{dT}{dt}$ (remember, T_s is constant). Substituting $\frac{dT}{dt} = \frac{dy}{dt}$ and $y = T - T_s$ into the Newton Law of cooling model gives the equation

$$\frac{dy}{dt} = k\, y\,.$$

This is just the basic exponential growth model. The solution of this differential equation is

$$y(t) = y_0 e^{kt}\,,$$

where y_0 is the initial value of $y(t)$ at time $t = 0$, that is $y(0) = y_0$. We last need to change to solution back into an equation involving the temperature *T*. Recall that $y(t) = T(t) - T_s$. Using this equation, we see that $y_0 = y(0) = T(0) - T_s = T_0 - T_s$ (we use the variable T_0 to represent the initial temperature of the object at time $t = 0$, that is $T(0) = T_0$. Substituting $y(t) = T(t) - T_s$ and $y_0 = T_0 - T_s$ into $y(t) = y_0 e^{kt}$ gives

$$T(t) - T_s = (T_0 - T_s)e^{kt}$$

Solving for $T(t)$ gives the solution to the Newton Law of Cooling differential equation:

$$T(t) = T_s + (T_0 - T_s)e^{kt}$$

We summarize the result as follows:

Newton's Law of Cooling

The rate that the temperature T of an object that is cooling is given by the initial value problem

$$\frac{dT}{dt} = k(T - T_s), \quad T(0) = T_0.$$

The particular solution of this initial value problem describing the objects temperature after a particular time t is given by

$$T(t) = T_s + (T_0 - T_s)e^{kt}$$

where

$T_s =$ the temperature of the surrounding environment.
$T_0 =$ the initial temperature of the object at time $t = 0$.
$k =$ proportionality constant
$T(t)$ = the temperature of the object after time t.

We illustrate how this law works in the next example.

Example 5: Suppose you cool a pot of soup in a 75^0 F room. Right when you take the soup off the stove, you measure its temperature to be 220^0 F. Suppose after 20 minutes, the soup has cooled to 170^0 F.

a. What will be the temperature of the soup in 30 minutes.
b. Suppose you can eat the soup when it is 130^0 F. How long will it take to cool to this temperature?

Solution: (Part a) The solution to initial value problem $\frac{dT}{dt} = k(T - T_s),\ \ T(0) = T_0$ describing Newton's Law of Cooling is

$$T(t) = T_s + (T_0 - T_s)e^{kt}$$

Since the temperature of the room is 75^0, $T_s = 75$. Since the soup when taken off the stove is 220^0, $T_0 = 220$. Thus, the formula for describing the temperature after time *t* is given by

$$T(t) = 75 + (220 - 75)e^{kt}$$

or, when simplified,

$$T(t) = 75 + 145e^{kt}$$

As with any exponential model, if the proportionality constant *k*, is not given, we must find it. To do this, we use the fact that after *t* = 20 minutes, the temperature is 170^0. Mathematically, this says that $T(20) = 170$. We substitute this fact into the above equation and solve for *k* using the following steps.

$$170 = T(20) = 75 + 145e^{k(20)}$$

$$75 + 145e^{20k} = 170 \quad \text{(Rearrange and clean up the resulting equation)}$$

$$145e^{20k} = 170 - 75 \quad \text{(Substract 75 from both sides)}$$

$$145e^{20k} = 95 \quad \text{(Simplify)}$$

$$e^{20k} = \frac{95}{145} \quad \text{(Divide both sides by 145)}$$

$$\ln e^{20k} = \ln\left(\frac{95}{145}\right) \quad \text{(Take ln of both sides)}$$

$$20k \ln e = \ln\left(\frac{95}{145}\right) \quad \text{(Use property } \ln b^u = u \ln b\text{)}$$

$$20k(1) = \ln\left(\frac{95}{145}\right) \quad (\ln e = 1)$$

$$k = \frac{\ln(95/145)}{20} \quad \text{(Divide both sides by 20 to solve for } k\text{)}$$

$$k \approx -0.0211 \quad \text{(Use calculator)}$$

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Substituting $k = -0.0211$ into $T(t) = 75 + 145e^{kt}$ gives

$$T(t) = 75 + 145e^{-0.0211t}$$

Thus, we find the temperature after $t = 30$ substituting into this equation. Hence,

$$\begin{array}{c}\text{Temperature of Soup} \\ \text{after } t = 30 \text{ minutes}\end{array} = T(30) = 75 + 145e^{-0.0211(30)}$$

$$= 75 + 145e^{-0.633}$$

$$\boxed{= 152^0 F}$$

Part b) We want the time t it takes for the soup to cool to 130^0, that is, the time t when $T(t) = 130$. Taking the equation we found in part a $T(t) = 75 + 145e^{-0.0211t}$, we solve this equation as follows:

$130 = T(t) = 75 + 145e^{-0.0211t}$

$75 + 145e^{-0.0211t} = 130$ (Rearrange the resulting equation)

$145e^{-0.0211t} = 130 - 75$ (Substract 75 from both sides)

$145e^{-0.0211t} = 55$ (Simplify)

$e^{-0.0211t} = \dfrac{55}{145}$ (Divide both sides by 145)

$\ln e^{-0.0211t} = \ln\left(\dfrac{55}{145}\right)$ (Take ln of both sides)

$-0.0211t \ln e = \ln\left(\dfrac{55}{145}\right)$ (Use property $\ln b^u = u \ln b$)

$-0.0211t\,(1) = \ln\left(\dfrac{55}{145}\right)$ ($\ln e = 1$)

$t = \dfrac{\ln(55/145)}{-0.0211}$ (Divide both sides by -0.0211 to solve for t)

$t \approx 46$ (Use calculator)

Thus, it takes approximately $\boxed{t = 46 \text{ minutes}}$ for the soup to cool to $130^0 F$.

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